

Incomputability of Simply Connected Planar Continua

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Abstract

Le Roux and Ziegler asked whether every simply connected compact nonempty planar Π_1^0 set always contains a computable point. In this paper, we solve the problem of le Roux and Ziegler by showing that there exists a planar Π_1^0 dendroid without computable points. We also provide several pathological examples of tree-like Π_1^0 continua fulfilling certain global incomputability properties: there is a computable dendrite which does not $*$ -include a Π_1^0 tree; there is a Π_1^0 dendrite which does not $*$ -include a computable dendrite; there is a computable dendroid which does not $*$ -include a Π_1^0 dendrite. Here, a continuum A $*$ -includes a member of a class \mathcal{P} of continua if, for every positive real ε , A includes a continuum $B \in \mathcal{P}$ such that the Hausdorff distance between A and B is smaller than ε .

1 Background

Every nonempty open set in a computable metric space (such as Euclidean n -space \mathbb{R}^n) contains a computable point. In contrast, the Non-Basis Theorem asserts that a nonempty *co-c.e. closed* set (also called a Π_1^0 set) in Cantor space (hence, even in Euclidean 1-space) can avoid any computable points. Non-Basis Theorems can shed new light on connections between *local* and *global* properties by incorporating the notions of *measure* and *category*. For instance, Kreisel-Lacombe [6] and Tanaka [17] showed that there is a Π_1^0 set with positive measure that contains no computable point. Recent exciting progress in *Computable Analysis* [18] naturally raises the question whether Non-Basis Theorems exist for *connected* Π_1^0 sets. However, we observe that, if a nonempty Π_1^0 subset of \mathbb{R}^1 contains no computable points, then it must be totally disconnected. Then, in higher dimensional Euclidean space, can there exist a connected Π_1^0 set containing no computable points? It is easy to construct a nonempty connected Π_1^0 subset of $[0, 1]^2$ without computable points, and a nonempty simply connected Π_1^0 subset of $[0, 1]^3$ without computable points. An open problem, formulated by Le Roux and Ziegler [13] was whether every nonempty simply connected compact planar Π_1^0 set contains a computable point. As mentioned in Penrose's book "*Emperor's New Mind*" [12], *the Mandelbrot set* is an example of a simply connected compact planar Π_1^0 set which contains a computable point, and he conjectured that the Mandelbrot set is not computable *as a closed set*. Hertling [5] observed that the Penrose conjecture has an implication for a

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famous open problem on local connectivity of the Mandelbrot set. Our interest is which topological assumption (especially, connectivity assumption) on a Π_1^0 set can force it to possess a given computability property. Miller [10] showed that every Π_1^0 sphere in \mathbb{R}^n is computable, and so it contains a dense c.e. subset of computable points. He also showed that every Π_1^0 ball in \mathbb{R}^n contains a dense subset of computable points. Iljazović [7] showed that chainable continua (e.g., arcs) in certain metric spaces are almost computable, and hence there always is a dense subset of computable points. In this paper, we show that *not* every Π_1^0 dendrite is almost computable, by using a tree-immune Π_1^0 class in Cantor space. This notion of immunity was introduced by Cenzer, Weber Wu, and the author [4]. We also provide pathological examples of tree-like Π_1^0 continua fulfilling certain global incomputability properties: there is a computable dendrite which does not $*$ -include a Π_1^0 tree; there is a computable dendroid which does not $*$ -include a Π_1^0 dendrite. Finally, we solve the problem of Le Roux and Ziegler [13] by showing that there exists a planar Π_1^0 dendroid without computable points. Indeed, our planar dendroid is contractible. Hence, our dendroid is also the first example of a contractible Euclidean Π_1^0 set without computable points.

2 Preliminaries

Basic Notation: $2^{<\mathbb{N}}$ denotes the set of all finite binary strings. Let X be a topological space. For a subset $Y \subseteq X$, $cl(Y)$ ($int(Y)$, resp.) denotes the closure (the interior, resp.) of Y . Let $(X; d)$ be a metric space. For any $x \in X$ and $r \in \mathbb{R}$, $B(x; r)$ denotes the open ball $B(x; r) = \{y \in X : d(x, y) < r\}$. Then x is called *the center of $B(x; r)$* , and r is called *the radius of $B(x; r)$* . For a given open ball $B = B(x; r)$, \bar{B} denotes the corresponding closed ball $\bar{B} = \{y \in X : d(x, y) \leq r\}$. For $a, b \in \mathbb{R}$, $[a, b]$ denotes the closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, (a, b) denotes the open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, and $\langle a, b \rangle$ denotes a point of Euclidean plane \mathbb{R}^2 . For $X \subseteq \mathbb{R}^n$, $\text{diam}(X)$ denotes $\max\{d(x, y) : x, y \in X\}$.

Continuum Theory: A *continuum* is a compact connected metric space. For basic terminology concerning *Continuum Theory*, see Nadler [11] and Illanes-Nadler [8].

Let X be a topological space. The set X is a *Peano continuum* if it is a locally connected continuum. The set X is a *dendrite* if it is a Peano continuum which contains no Jordan curve. The set X is *unicoherent* if $A \cap B$ is connected for every connected closed subsets $A, B \subseteq X$ with $A \cup B = X$. The set X is *hereditarily unicoherent* if every subcontinuum of X is unicoherent. The set X is a *dendroid* if it is an arcwise connected hereditary unicoherent continuum. For a point x of a dendroid X , $r_X(x)$ denotes the cardinality of the set of arc-components of $X \setminus \{x\}$. If $r_X(x) \geq 3$ then x is said to be a *ramification point of X* . The set X is a *tree* if it is dendrite with finitely many ramification points. Note that a topological space X is a dendrite if and only if it is a locally connected dendroid. Hahn-Mazurkiewicz's Theorem states that a Hausdorff space X is a Peano continuum if and only if X is an image of a continuous curve.

Example 1 (Planar Dendroids).

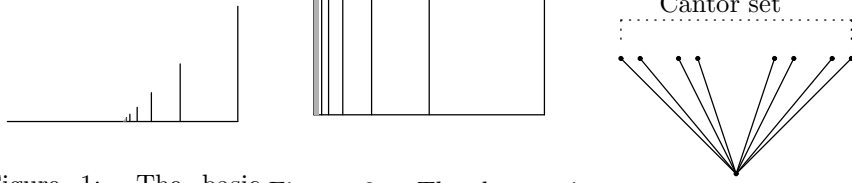


Figure 1: The basic dendrite Figure 2: The harmonic comb Figure 3: The Cantor fan

1. Put $\mathcal{B}_t = \{2^{-t}\} \times [0, 2^{-t}]$. Then the following set $\mathcal{B} \subseteq \mathbb{R}^2$ is dendrite.

$$\mathcal{B} = \bigcup_{t \in \mathbb{N}} \mathcal{B}_t \cup ([-1, 1] \times \{0\}).$$

We call \mathcal{B} the *basic dendrite*. The set \mathcal{B}_t is called the *t-th rising of \mathcal{B}* . See Fig. 1.

2. The set $\mathcal{H} = cl((\{1/n : n \in \mathbb{N}\} \times [0, 1]) \cup ([0, 1] \times \{0\}))$ is called a *harmonic comb*. Then \mathcal{H} is a dendroid, but not a dendrite. The set $\{1/n\} \times [0, 1]$ is called the *n-th rising of the comb \mathcal{H}* , and the set $[0, 1] \times \{0\}$ is called the *grip of \mathcal{H}* . See Fig. 2.
3. Let $C \subseteq \mathbb{R}^1$ be the middle third Cantor set. Then the one-point compactification of $C \times (0, 1]$ is called the *Cantor fan*. (Equivalently, it is the quotient space $\text{Cone}(C) = (C \times [0, 1]) / (C \times \{0\})$.) The Cantor fan is a dendroid, but not a dendrite. See Fig. 3.

Let X be a topological space. X is *n-connected* if it is path-connected and $\pi_i(X) \equiv 0$ for any $1 \leq i \leq n$, where $\pi_i(X)$ is the *i*-th homotopy group of X . X is *simply connected* if X is 1-connected. X is *contractible* if the identity map on X is null-homotopic. Note that, if X is contractible, then X is *n-connected* for each $n \geq 1$. It is easy to see that the dendroids in Example 1 are contractible.

Computability Theory: We assume that the reader is familiar with Computability Theory on the natural numbers \mathbb{N} , Cantor space $2^{\mathbb{N}}$, and Baire space $\mathbb{N}^{\mathbb{N}}$ (see also Soare [16]). For basic terminology concerning *Computable Analysis*, see Weihrauch [18], Brattka-Weihrauch [3], and Brattka-Presser [2].

Hereafter, we fix a countable base for the Euclidean n -space \mathbb{R}^n by $\rho = \{B(x; r) : x \in \mathbb{Q}^n \text{ \& } r \in \mathbb{Q}^+\}$, where \mathbb{Q}^+ denotes the set of all positive rationals. Let $\{\rho_n\}_{n \in \mathbb{N}}$ be an effective enumeration of ρ . We say that a point $x \in \mathbb{R}^n$ is *computable* if the code of its principal filter $\mathcal{F}(x) = \{i \in \mathbb{N} : x \in \rho_i\}$ is computably enumerable (hereafter c.e.) A closed subset $F \subseteq \mathbb{R}^n$ is Π_1^0 if there is a c.e. set $W \subseteq \mathbb{N}$ such that $F = X \setminus \bigcup_{e \in W} \rho_e$. A closed subset $F \subseteq \mathbb{R}^n$ is *computably enumerable* (hereafter c.e.) if $\{e \in \mathbb{N} : F \cap \rho_e \neq \emptyset\}$ is c.e. A closed subset $F \subseteq \mathbb{R}^n$ is *computable* if it is Π_1^0 and c.e. on \mathbb{R}^n .

Almost Computability: Let A_0, A_1 be nonempty closed subsets of a metric space (X, d) . Then the *Hausdorff distance* between A_0 and A_1 is defined by

$$d_H(A_0, A_1) = \max_{i < 2} \sup_{x \in A_i} \inf_{y \in A_{1-i}} d(x, y).$$

Let \mathcal{P} be a class of continua. We say that a continuum A **-includes a member of \mathcal{P}* if $\inf\{d_H(A, B) : A \supseteq B \in \mathcal{P}\} = 0$.

Proposition 2. *Every Euclidean dendroid \ast -includes a tree.*

Proof. Fix a Euclidean dendroid $D \subseteq \mathbb{R}^n$, and a positive rational $\varepsilon \in \mathbb{Q}$. Then D is covered by finitely many open rational balls $\{B_i\}_{i < n}$ of radius $\varepsilon/2$. Choose $d_i \in D \cap B_i$ for each $i < n$ if B_i intersects with D . Note that $\{B(d_i; \varepsilon)\}_{i < n}$ covers D . Since D is dendroid, there is a unique arc $\gamma_{i,j} \subseteq D$ connecting d_i and d_j for each $i, j < n$. Then, $E = \bigcup_{\{i,j\} \subseteq n} \gamma_{i,j}$ is connected and locally connected, since E is a union of finitely many arcs (i.e., it is a graph, in the sense of Continuum Theory; see also Nadler [11]). It is easy to see that E has no Jordan curve, since E is a subset of the dendroid D . Consequently, E is a tree. Moreover, clearly $d_H(E, D) < \varepsilon$, since $d_i \in E$ for each $i < n$. \square

The class \mathcal{P} has the *almost computability property* if every $A \in \mathcal{P}$ \ast -includes a computable member of \mathcal{P} as a closed set. In this case, we simply say that *every $A \in \mathcal{P}$ is almost computable*. Iljazović [7] showed that every Π_1^0 chainable continuum is almost computable, hence every Π_1^0 arc is almost computable.

3 Incomputability of Dendrites

By Proposition 2, topologically, every planar dendrite \ast -includes a tree. However, if we try to effectivize this fact, we will find a counterexample.

Theorem 3. *Not every computable planar dendrite \ast -includes a Π_1^0 tree.*

Proof. Let $A \subseteq \mathbb{N}$ be an incomputable c.e. set. Thus, there is a total computable function $f_A : \mathbb{N} \rightarrow \mathbb{N}$ such that $\text{range}(f_A) = A$. We may assume $f_A(s) \leq s$ for every $s \in \mathbb{N}$. Let A_s denote the finite set $\{f_A(u) : u \leq s\}$. Then $\text{st}^A : \mathbb{N} \rightarrow \mathbb{N}$ is defined as $\text{st}^A(n) = \min\{s \in \mathbb{N} : n \in A_s\}$. Note that $\text{st}^A(n) \geq n$ by our assumption $f_A(s) \leq s$.

Construction. Recall the definition of the basic dendrite from Example 1. We construct a computable dendrite by modifying the basic dendrite \mathcal{B} . For every $t \in \mathbb{N}$, we introduce the *width of the t -rising* $w(t)$ as follows:

$$w(t) = \begin{cases} 2^{-(2+\text{st}^A(t))}, & \text{if } t \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Let I_t be the closed interval $[2^{-t} - w(t), 2^{-t} + w(t)]$. Since $\text{st}^A(n) \geq n$, we have $I_t \cap I_s = \emptyset$ whenever $t \neq s$. We observe that $\{w(t)\}_{t \in \mathbb{N}}$ is a uniformly computable sequence of real numbers. Now we define a computable dendrite $D \subseteq \mathbb{R}^2$ by:

$$\begin{aligned} D_t^0 &= (\{2^{-t} - w(t)\} \cup \{2^{-t} + w(t)\}) \times [0, 2^{-t}] \\ D_t^1 &= [2^{-t} - w(t), 2^{-t} + w(t)] \times \{2^{-t}\} \\ D_t^2 &= (2^{-t} - w(t), 2^{-t} + w(t)) \times (-1, 2^{-t}) \\ D &= \left(\bigcup_{t \in \mathbb{N}} (D_t^0 \cup D_t^1) \right) \cup \left(([-1, 1] \times \{0\}) \setminus \bigcup_{t \in \mathbb{N}} D_{t,m}^2 \right). \end{aligned}$$

We call $D_t = D_t^0 \cup D_t^1$ the t -th rising of D . See Fig. 4.

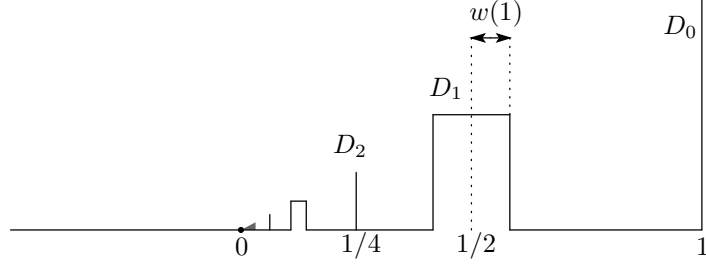


Figure 4: The dendrite D for $0, 2, 4 \notin A$ and $1, 3 \in A$.

Claim. The set D is a dendrite.

To prove D is a Peano continuum, by the Hahn-Mazurkiewicz Theorem, it suffices to show that $D = \text{Im}(h)$ for some continuous curve $h : [-1, 1] \rightarrow \mathbb{R}^2$. We divide the unit interval $[0, 1]$ into infinitely many parts $I_t = [2^{-(t+1)}, 2^{-t}]$. Furthermore, we also divide each interval I_{2t} into three parts I_{2t}^0, I_{2t}^1 , and I_{2t}^2 , where $I_{2t}^i = [(5-i) \cdot 3^{-1} \cdot 2^{-(2t+1)}, (6-i) \cdot 3^{-1} \cdot 2^{-(2t+1)}]$ for each $i < 3$. Then we define a desired curve h as follows.

$$h(x) \text{ moves in } \begin{cases} \{2^{-t} + w(t)\} \times [0, 2^{-t}] & \text{if } x \in I_{2t}^0, \\ [2^{-t} - w(t), 2^{-t} + w(t)] \times \{2^{-t}\} & \text{if } x \in I_{2t}^1, \\ \{2^{-t} - w(t)\} \times [0, 2^{-t}] & \text{if } x \in I_{2t}^2, \\ [2^{-(t+1)} + w(t+1), 2^{-t} - w(t)] \times \{0\} & \text{if } x \in I_{2t+1}, \\ [-1, 0] \times \{0\} & \text{if } x \in [-1, 0]. \end{cases}$$

Clearly, h can be continuous, and indeed computable, since the map $w : \mathbb{R} \rightarrow \mathbb{R}$ is computable. It is easy to see that $D = \text{Im}(h)$. Moreover, $\text{Im}(h)$ contains no Jordan curve since $\pi_0(h(x)) \leq \pi_0(h(y))$ whenever $x \leq y$, where $\pi_0(p)$ denotes the first coordinate of $p \in \mathbb{R}^2$. Consequently, D is a dendrite.

Moreover, by construction, it is easy to see that D is computable.

Claim. The computable dendrite D does not $*$ -include a Π_1^0 tree.

Suppose that D contains a Π_1^0 subtree $T \subseteq D$. We consider a rational open ball B_t with center $\langle 2^{-t}, 2^{-t} \rangle$ and radius $2^{-(t+2)}$, for each $t \in \mathbb{N}$. Note that $B_t \cap D \subseteq D_t$ for every $t \in \mathbb{N}$. Since T is Π_1^0 in \mathbb{R}^2 , $B = \{t \in \mathbb{N} : \hat{B}_t \cap T = \emptyset\}$ is c.e. If $w(t) > 0$ (i.e., $t \in A$) then $D \setminus (D_t \cap B_t)$ is disconnected. Therefore, either $T \subseteq [-1, 2^{-t}] \times \mathbb{R}$ or $T \subseteq [2^{-t}, 1] \times \mathbb{R}$ holds whenever $\hat{B}_t \cap T = \emptyset$ (i.e., $t \in B$), since T is connected. Thus, if the condition $\#(A \cap B) = \aleph_0$ is satisfied, then either $T \subseteq [-1, 0] \times \mathbb{R}$ or $T \subseteq [0, 1] \times \mathbb{R}$ holds. Consequently, we must have $d_H(T, D) \geq 1$.

Therefore, we may assume $\#A \cap B < \aleph_0$. Since A is coinfinite, D has infinitely many ramification points $\langle 2^{-t}, 0 \rangle$ for $t \notin A$. However, by the definition of tree, T has only finitely many ramification points. Thus we must have $(D_t^0 \cap T) \setminus \{\langle 2^{-t}, 0 \rangle\} = \emptyset$ for almost all $t \notin A$. Since $\hat{B}_t \cap T \subseteq (D_t^0 \cap T) \setminus \{\langle 2^{-t}, 0 \rangle\}$, we have $t \in B$ for almost all $t \in \mathbb{N} \setminus A$. Consequently, we have $\#((\mathbb{N} \setminus A) \triangle B) < \aleph_0$. This implies that $\mathbb{N} \setminus A$ is also c.e., since B is c.e. This contradicts that A is incomputable. \square

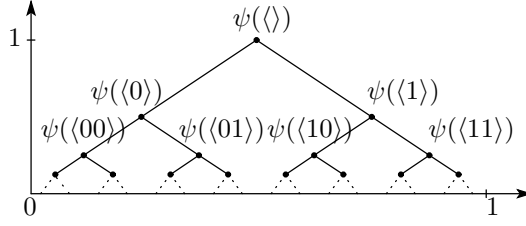


Figure 5: The plotted tree $\Psi(2^{<\mathbb{N}})$.

Note that a Hausdorff space (hence every metric space) is (locally) arcwise connected if and only if it is (locally) pathwise connected. However, Miller [10] pointed out that the effective versions of arcwise connectivity and pathwise connectivity do *not* coincide. Theorem 3 could give a result on effective connectivity properties. Note that *effectively pathwise connectivity* is defined by Brattka [1]. Clearly, the dendrite D is effectively pathwise connected. We now introduce a new effective version of arcwise connectivity property by thinking arcs as closed sets. Let $\mathcal{A}_-(X)$ denote the hyperspace of closed subsets of X with negative information (see also Brattka [1]).

Definition 4. A computable metric space (X, d, α) is *semi-effectively arcwise connected* if there exists a total computable multi-valued function $P : X^2 \rightrightarrows \mathcal{A}_-(X)$ such that $P(x, y)$ is the set of all arcs A whose two end points are x and y , for any $x, y \in X$.

Obviously D is not semi-effectively arcwise connected. Indeed, for every $\varepsilon > 0$ there exists $x_0, x_1 \in [0, 1]$ with $d(x_0, x_1) < \varepsilon$ such that $\langle x_0, 0 \rangle, \langle x_1, 0 \rangle \in D$ cannot be connected by any Π_1^0 arc. Thus, we have the following corollary.

Corollary 1. *There exists an effectively pathwise connected Euclidean continuum D such that D is not semi-effectively arcwise connected.*

Theorem 5. *Not every Π_1^0 planar dendrite is almost computable.*

To prove Theorem 5, we need to prepare some tools. For a string $\sigma \in 2^{<\mathbb{N}}$, let $lh(\sigma)$ denote the length of σ . Then

$$\psi(\sigma) = \left\langle 2^{-1} \cdot 3^{-i} + 2 \sum_{i < lh(\sigma) \text{ \& } \sigma(i)=1} 3^{-(i+1)}, 2^{-lh(\sigma)} \right\rangle \in \mathbb{R}^2.$$

For two points $\vec{x}, \vec{y} \in \mathbb{R}^2$, the closed line segment $L(\vec{x}, \vec{y})$ from \vec{x} to \vec{y} is defined by $L(\vec{x}, \vec{y}) = \{(1-t)\vec{x} + t\vec{y} : t \in [0, 1]\}$. For a (possibly infinite) tree $T \subseteq 2^{<\mathbb{N}}$, we plot an embedded tree $\Psi(T) \subseteq \mathbb{R}^2$ by

$$\Psi(T) = cl \left(\bigcup \{L(\psi(\sigma), \psi(\tau)) : \sigma, \tau \in T \text{ \& } lh(\sigma) = lh(\tau) + 1\} \right).$$

Then $\Psi(T)$ is a dendrite (but not necessarily a tree, in the sense of Continuum Theory), for any (possibly infinite) tree $T \subseteq 2^{<\mathbb{N}}$. See Fig. 5.

We can easily prove the following lemmata.

Lemma 6. *Let T be a subtree of $2^{<\mathbb{N}}$, and D be a planar subset such that $\psi(\langle \rangle) \in D \subseteq \Psi(T)$ for the root $\langle \rangle \in 2^{<\mathbb{N}}$. Then D is a dendrite if and only if D is homeomorphic to $\Psi(S)$ for a subtree $S \subseteq T$.*

Proof. The “if” part is obvious. Let D be a dendrite. For a binary string σ which is not a root, let σ^- be an immediate predecessor of σ . We consider the set $S = \{\langle \rangle\} \cup \{\sigma \in 2^{<\mathbb{N}} : \sigma \neq \langle \rangle \text{ \& } D \cap (L(\psi(\sigma^-), \psi(\sigma)) \setminus \{\psi(\sigma^-)\}) \neq \emptyset\}$. Since D is connected, S is a subtree of T . It is easy to see that D is homeomorphic to $\Psi(S)$. \square

Lemma 7. *Let T be a subtree of $2^{<\mathbb{N}}$. Then T is Π_1^0 (c.e., computable, resp.) if and only if $\Psi(T)$ is a Π_1^0 (c.e., computable, resp.) dendrite in \mathbb{R}^2 .*

Proof. With our definition of Ψ , the dendrite $\Psi(2^{<\mathbb{N}})$ is clearly a computable closed subset of \mathbb{R}^2 . So, if T is Π_1^0 , then it is easy to prove that $\Psi(T)$ is also Π_1^0 . Assume that T is a c.e. tree. At stage s , we compute whether $L(\psi(\sigma^-), \psi(\sigma))$ intersects with the e -th open rational ball ρ_e , for any $e < s$ and any string σ which is already enumerated into T by stage s . If so, we enumerate e into W_T at stage s . Then $\{e \in \mathbb{N} : \Psi(T) \cap \rho_e \neq \emptyset\} = W_T$ is c.e.

Assume that $\Psi(T)$ is Π_1^0 . We consider an open rational ball $B_-(\sigma) = B(\psi(\sigma); 2^{-(lh(\sigma)+2)})$ for each $\sigma \in 2^{<\mathbb{N}}$. Note that $\hat{B}_-(\sigma) \cap \hat{B}_-(\tau) = \emptyset$ for $\sigma \neq \tau$. Since $\Psi(T)$ is Π_1^0 , $T^* = \{\sigma \in 2^{<\mathbb{N}} : \Psi(T) \cap \hat{B}_-(\sigma) = \emptyset\}$ is c.e., and it is easy to see that $T = 2^{<\mathbb{N}} \setminus T^*$. Thus, T is a Π_1^0 tree of $2^{<\mathbb{N}}$. We next assume that $\Psi(T)$ is c.e. We can assume that $\Psi(T)$ contains the root $\psi(\langle \rangle)$, otherwise $T = \emptyset$, and clearly it is c.e. For a binary string σ which is not a root, let σ^- be an immediate predecessor of σ . Pick an open rational ball $B_+(\sigma)$ such that $\Psi(2^{<\mathbb{N}}) \cap B_+(\sigma) \subseteq L(\psi(\sigma^-), \psi(\sigma))$ for each σ . Since $\Psi(T)$ is c.e., $T^* = \{\sigma \in 2^{<\mathbb{N}} : \Psi(T) \cap B_+(\sigma) \neq \emptyset\}$ is c.e. If σ is not a root and $\sigma \in T$ then $L(\psi(\sigma^-), \psi(\sigma)) \subseteq \Psi(T)$, so $\Psi(T) \cap B_+(\sigma) \neq \emptyset$. We observe that if $\sigma \notin T$ then $L(\psi(\sigma^-), \psi(\sigma)) \cap \Psi(T) = \emptyset$, so $\Psi(T) \cap B_+(\sigma) = \emptyset$. Thus, we have $T = T^*$. In the case that $\Psi(T)$ is computable, $\Psi(T)$ is c.e. and Π_1^0 , hence T is c.e. and Π_1^0 , i.e., T is computable. \square

Lemma 8. *Let D be a computable subdendrite of $\Psi(2^{<\mathbb{N}})$. Then there exists a computable subtree $T^+ \subseteq 2^{<\mathbb{N}}$ such that $D \subseteq \Psi(T^+)$ and $([0, 1] \times \{0\}) \cap D = ([0, 1] \times \{0\}) \cap \Psi(T^+)$.*

Proof. We can assume $\psi(\langle \rangle) \in D$, otherwise we connect $\psi(\langle \rangle)$ and the root of D by a subarc of $\Psi(2^{<\mathbb{N}})$. Again we consider an open rational ball $B_-(\sigma) = B(\psi(\sigma); 2^{-(lh(\sigma)+2)})$, and an open rational ball $B_+(\sigma)$ such that $\Psi(2^{<\mathbb{N}}) \cap B_+(\sigma) \subseteq L(\psi(\sigma^-), \psi(\sigma))$ for each $\sigma \in 2^{<\mathbb{N}}$. Since D is Π_1^0 , $U^* = \{\sigma \in 2^{<\mathbb{N}} : D \cap \hat{B}_-(\sigma) = \emptyset\}$ is c.e. Since D is c.e., $T^* = \{\sigma \in 2^{<\mathbb{N}} : D \cap B_+(\sigma) \neq \emptyset\}$ is c.e., and it is a tree by Lemma 6. For every $\sigma \in 2^{<\mathbb{N}}$, either $D \cap \hat{B}_-(\sigma) = \emptyset$ or $D \cap B_+(\sigma) \neq \emptyset$ holds. Therefore, we have $T^* \cup U^* = 2^{<\mathbb{N}}$. Moreover, for the set of leaves of T^* , $L_T^* = \{\rho \in T^* : (\forall i < 2) \rho \frown \langle i \rangle \notin T^*\}$, we observe that $T^* \cap U^* \subseteq L_T^*$. Recall that the pointclass Σ_1^0 has the reduction property, that is, for two c.e. sets T^* and U^* , there exist c.e. subsets $T \subseteq T^*$ and $U \subseteq U^*$ such that $T \cup U = T^* \cup U^*$ and $T \cap U = \emptyset$. This is because, for $\sigma \in T^* \cap U^*$, σ is enumerated into T when σ is enumerated into T^* before it is enumerated into U^* ; σ is enumerated into U otherwise. Since $T^* \cap U^* \subseteq L_T^*$, T must be tree. Furthermore, T is c.e., and $U = 2^{<\mathbb{N}} \setminus T$ is also c.e. Thus, T is a computable tree. Therefore, $T^+ = \{\sigma \frown \langle i \rangle : \sigma \in T \text{ \& } i < 2\}$ is also a computable tree. Then, $D \subseteq \Psi(T^+)$, and we have $([0, 1] \times \{0\}) \cap D = ([0, 1] \times \{0\}) \cap \Psi(T^+)$ since the set of all infinite paths of T and that of T^+ coincide. \square

Cenzer, Weber and Wu, and the author [4] introduced the notion of *tree-immunity* for closed sets in Cantor space $2^{\mathbb{N}}$. For $\sigma \in 2^{<\mathbb{N}}$, define I_σ as $\{f \in 2^{\mathbb{N}} : (\forall n < lh(\sigma)) f(n) = \sigma(n)\}$. Note that $\{I_\sigma : \sigma \in 2^{<\mathbb{N}}\}$ is a countable base for Cantor space.

Definition 9 (Cenzer-Kihara-Weber-Wu [4]). A nonempty closed set $F \subseteq 2^{\mathbb{N}}$ is said to be *tree-immune* if the tree $T_F = \{\sigma \in 2^{<\mathbb{N}} : F \cap I_\sigma \neq \emptyset\} \subseteq 2^{<\mathbb{N}}$ contains no infinite computable subtree.

For a nonempty Π_1^0 subset $P \subseteq 2^{\mathbb{N}}$, the corresponding tree T_P is Π_1^0 , and it has no dead ends. The set of *all complete consistent extensions of Peano Arithmetic* is an example of a tree-immune Π_1^0 subset of $2^{\mathbb{N}}$. Tree-immune Π_1^0 sets have the following remarkable property.

Lemma 10. *Let P be a tree-immune Π_1^0 subset of $2^{\mathbb{N}}$ and let $D \subseteq \Psi(T_P)$ be any computable subdendrite. Then $([0, 1] \times \{0\}) \cap D = \emptyset$ holds.*

Proof. By Lemma 8, there exists a computable subtree $T \subseteq 2^{<\mathbb{N}}$ such that $D \subseteq \Psi(T)$ and $\Psi(T)$ agrees with D on $[0, 1] \times \{0\}$. Since $D \subseteq \Psi(T_P)$, and since T_P has no dead ends, $T \subseteq T_P$ holds. Since P is tree-immune, T must be finite. By using weak König's lemma (or, equivalently, compactness of Cantor space), $T \subseteq 2^l$ holds for some $l \in \mathbb{N}$. Thus, $D \subseteq \Psi(T) \subseteq [0, 1] \times [2^{-l}, 1]$ as desired. \square

Note that if P is a nonempty Π_1^0 set in Cantor space $2^{\mathbb{N}}$, then for every $\delta > 0$ it holds that $((0, 1) \times (0, \delta)) \cap \Psi(T_P) \neq \emptyset$. Finally, we are ready to prove Theorem 5.

Proof of Theorem 5. Again, we adapt the construction in the proof of Theorem 3. We fix a nonempty tree-immune Π_1^0 set $P \subseteq 2^{\mathbb{N}}$. For $\sigma \in 2^{<\mathbb{N}}$, put $E(\sigma) = \{\tau \in 2^{<\mathbb{N}} : \tau \supseteq \sigma\}$. For a Π_1^0 tree $T_P \subseteq 2^{<\mathbb{N}}$, there exists a computable function $f_P : \mathbb{N} \rightarrow 2^{<\mathbb{N}}$ such that $T_P = 2^{<\mathbb{N}} \setminus \bigcup_n E(f_P(n))$ and such that for each $\sigma \in 2^{<\mathbb{N}}$ and $s \in \mathbb{N}$ we have $\sigma \in \bigcup_{t < s} E(f_P(t))$ whenever $\sigma \frown 0, \sigma \frown 1 \in \bigcup_{t < s} E(f_P(t))$. For such a computable function $f_P : \mathbb{N} \rightarrow 2^{<\mathbb{N}}$, we let $T_{P,s}$ denote $2^{<\mathbb{N}} \setminus \bigcup_{t < s} E(f_P(t))$. Then $T_{P,s}$ is a tree without dead ends, and $\{T_{P,s} : s \in \mathbb{N}\}$ is computable uniformly in s .

Construction. . Let \vec{e}_1 denote $\langle 1, 0 \rangle \in \mathbb{R}^2$. For a tree $T \subseteq 2^{<\mathbb{N}}$ and $w \in \mathbb{Q}$, we define $\Psi(T; w)$, the *edge of the fat approximation of the tree T of width w* , by

$$\Psi(T; w) = cl\left(\bigcup \left\{ L\left(\psi(\sigma) \pm (3^{-|\sigma|} \cdot w)\vec{e}_1, \psi(\tau) \pm (3^{-|\tau|} \cdot w)\vec{e}_1\right) : \pm \in \{-, +\} \ \& \ \sigma, \tau \in T \ \& \ lh(\sigma) = lh(\tau) + 1 \right\}\right).$$

If $\lim_s w_s = 0$ then we have $\lim_s \Psi(T; w_s) = \Psi(T)$. Moreover, if $\{w_s : s \in \mathbb{N}\}$ is a uniformly computable sequence of rational numbers, then $\{\Psi(T; w_s) : s \in \mathbb{N}\}$ is also a uniformly computable sequence of computable closed sets. Additionally, the set $\Psi(T; w, c, t, q)$, for a tree $T \subseteq 2^{<\mathbb{N}}$, for $w, c, q \in \mathbb{Q}$, and for $t \in \mathbb{N}$, is defined by

$$\Psi(T; w, c, t, q) = \left\{ \left\langle c + q \cdot \left(x - \frac{1}{2}\right), \frac{2-y}{2^{t+1}} \right\rangle \in \mathbb{R}^2 : \langle x, y \rangle \in \Psi(T; w) \right\}.$$

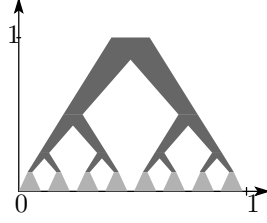


Figure 6: The fat approximation $\Psi(T; w)$.

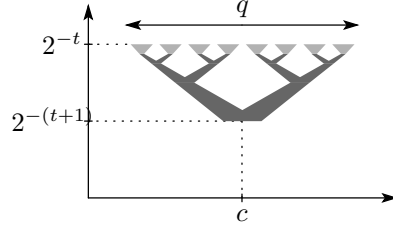


Figure 7: The basic object $\Psi(T; w, c, t, q)$.

Note that $\Psi(T; w, c, t, q) \subseteq [c - q/2, c + q/2] \times [2^{-(t+1)}, 2^{-t}]$ as in Fig. 7. For $t \in \mathbb{N}$, and for $\text{st}^A(t) = \min\{s : t \in A_s\}$ in the proof of Theorem 3, let $l(t) \in 2^{\mathbb{N}}$ be the leftmost path of $T_{P, \text{st}^A(t)}$. If $\text{st}^A(t)$ is undefined (i.e., $t \notin A$) then $l(t)$ is also undefined. For each $t \in \mathbb{N}$ we define $F(t) = \{\sigma \in 2^{<\mathbb{N}} : \sigma \subseteq l(t)\}$ if $l(t)$ is defined; $F(t) = T_P$ otherwise. Then $\{F(t) : t \in \mathbb{N}\}$ is a computable sequence of Π_1^0 subsets of $2^{<\mathbb{N}}$. Furthermore, we have $\Psi(F(t)) \cap ([0, 1] \times \{0\}) \neq \emptyset$, since $F(t)$ has a path for every $t \in \mathbb{N}$. For each $t \in \mathbb{N}$, $w(t)$ is defined again as in the proof of Theorem 3. Now we define a Π_1^0 dendrite $H \subseteq \mathbb{R}^2$ as follows:

$$\begin{aligned} H_t^* &= \Psi(F(t); w(t), 2^{-t}, t, 2^{-(t+2)}) \\ H_t^0 &= (\{2^{-t} - w(t)\} \cup \{2^{-t} + w(t)\}) \times [0, 2^{-(t+1)}] \\ H_t^{**} &= (2^{-t} - w(t), 2^{-t} + w(t)) \times \{2^{-(t+1)}\} \\ H_t^2 &= (2^{-t} - w(t), 2^{-t} + w(t)) \times (-1, 2^{-(t+1)}) \\ H &= \left(\bigcup_{t \in \mathbb{N}} (H_t^* \cup H_t^0 \setminus (H_t^{**} \cup \text{int} H_t^*)) \right) \cup \left(([-1, 1] \times \{0\}) \setminus \bigcup_{t \in \mathbb{N}} H_t^2 \right). \end{aligned}$$

Put $H_t = H_t^* \setminus (H_t^{**} \cup \text{int} H_t^*)$ (see Fig. 8). We can also show that H is a Π_1^0 dendrite in the same way as for Theorem 3.

Claim. The Π_1^0 dendrite H does not $*$ -include a computable dendrite.

Let J be a computable subdendrite of H . Put $S(t) = [3 \cdot 2^{-(t+2)}, 5 \cdot 2^{-(t+2)}] \times [2^{-(t+1)}, 2^{-t}]$. Then, we note that $J(t) = J \cap S(t)$ is also a computable dendrite, since $H_t \subseteq S(t)$ and it is a dendrite. However, by Lemma 10, if $t \notin A$ then we have $J(t) \cap (\mathbb{R} \times \{2^{-t}\}) = \emptyset$. So we consider the following set:

$$C = \{t \in \mathbb{N} : J(t) \cap ([3 \cdot 2^{-(t+2)}, 5 \cdot 2^{-(t+2)}] \times [2^{-t}, 1]) = \emptyset\}.$$

Since $J(t)$ is uniformly computable in t , the set C is clearly c.e., and we have $\mathbb{N} \setminus A \subseteq C$. However, if $\mathbb{N} \setminus A = C$, then this contradicts the incomputability of A . Thus, there must be infinitely many $t \in A$ such that t is enumerated into C . However, if $t \in A$ is enumerated into C , it *cuts* the dendrite H . In other words, since $J \subseteq H$ is connected, either $J \subseteq [-1, 5 \cdot 2^{-(t+2)}] \times \mathbb{R}$ or $J \subseteq [3 \cdot 2^{-(t+2)}, 1] \times \mathbb{R}$. Hence we must have $d_H(J, H) \geq 1$. \square

Corollary 2. *There exists a nonempty Π_1^0 subset of $[0, 1]^2$ which is contractible, locally contractible, and $*$ -includes no connected computable closed subset.*

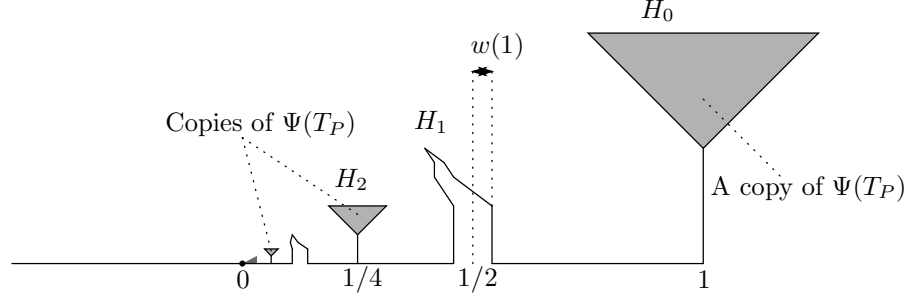


Figure 8: The dendrite H for $0, 2, 4 \notin A$ and $1, 3 \in A$.

4 Incomputability of Dendroids

Theorem 11. *Not every computable planar dendroid \ast -includes a Π_1^0 dendrite.*

Lemma 12. *There exists a limit computable function f such that, for every uniformly c.e. sequence $\{U_n : n \in \mathbb{N}\}$ of cofinite c.e. sets, we have $f(n) \in U_n$ for almost all $n \in \mathbb{N}$.*

Proof. Let $\{V_e : e \in \mathbb{N}\}$ be an effective enumeration of all uniformly c.e. non-increasing sequences $\{U_n : n \in \mathbb{N}\}$ of c.e. sets such that $\min U_n \geq n$, where $(V_e)_n = U_n = \{x \in \mathbb{N} : (n, x) \in V_e\}$. The e -state of y is defined by $\sigma(e, y) = \{i \leq e : y \in (V_i)_e\}$, and the maximal e -state is defined by $\sigma(e) = \max_z \sigma(e, z)$. The construction of $f : \mathbb{N} \rightarrow \mathbb{N}$ is to maximize the e -state. For each $e \in \mathbb{N}$, $f(e)$ chooses the least $y \in \mathbb{N}$ having the maximal e -state $\sigma(e, y) = \sigma(e)$. Since $\{\sigma(e, y) : e, y \in \mathbb{N}\}$ is uniformly c.e., and $\sigma(e, y) \in 2^e$, the function $e \mapsto \sigma(e) = \max_z \sigma(e, z)$ is total limit computable. Thus, f is limit computable. It is easy to see that $\lim_e \sigma(e)(n)$ exists for each $n \in \mathbb{N}$. Let $U = \{U_n : n \in \mathbb{N}\}$ be a uniformly c.e. sequence of cofinite c.e. sets. Then $V = \{\bigcap_{m \leq n} U_m : n \in \mathbb{N}\}$ is a uniformly c.e. non-increasing sequence of cofinite c.e. sets. Thus, $V_i = V$ for some index i . Then $i \in \sigma(e, y)$ for almost all $e, y \in \mathbb{N}$. This ensures that $i \in \sigma(e)$ for almost all $e \in \mathbb{N}$ by our assumption $\min U_n \geq n$. Hence we have $f(n) \in U_n$ for almost all $n \in \mathbb{N}$. \square

Remark. The proof of Lemma 12 is similar to the standard construction of a maximal c.e. set (see Soare [16]). Recall that the principal function of the complement of a maximal c.e. set is *dominant*, i.e., it dominates all total computable functions. The limit computable function f in Lemma 12 is also dominant. Indeed, for any total computable function g , if we set $U_n^g = \{y \in \mathbb{N} : y \geq g(n)\}$ then $\{U_n^g : n \in \mathbb{N}\}$ is a uniformly c.e. sequence of cofinite c.e. sets, and if $f(n) \in U_n^g$ holds then we have $f(n) \geq g(n)$.

Proof of Theorem 11. Pick a limit computable function $f = \lim_s f_s$ in Lemma 12. For every $t, u \in \mathbb{N}$, put $v(t, u) = 2^{-s}$ for the least s such that $f_s(t) = u$ if such s exists; $v(t, u) = 0$ otherwise. Since $\{f_s : s \in \mathbb{N}\}$ is uniformly computable, $v : \mathbb{N}^2 \rightarrow \mathbb{R}$ is computable.

Construction. . For each $t \in \mathbb{N}$, the center position of the u -th rising of the t -th comb is defined as $c_*(t, u) = 2^{-(2t+1)} + 2^{-(2t+u+1)}$, and the width of the

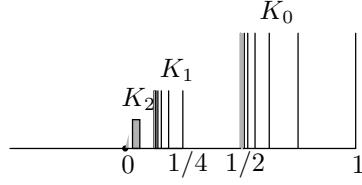


Figure 9: The dendroid K .

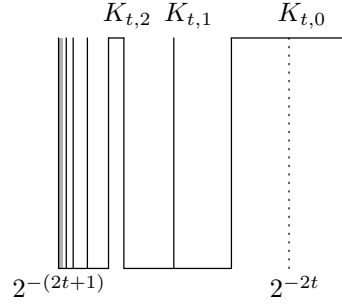


Figure 10: The harmonic comb K_t for $f_0(t) = 0, f_1(t) = 0, f_2(t) = 2, \dots$

u -th rising of the t -th comb is defined as $v_*(t, u) = v(t, u) \cdot 2^{-(2t+u+3)}$. Then, we define the t -th harmonic comb K_t for each $t \in \mathbb{N}$ as follows:

$$\begin{aligned} K_t^* &= \{2^{-(2t+1)}\} \times [0, 2^{-t}] \\ K_{t,u}^0 &= \{c_*(t, u) - v_*(t, u), c_*(t, u) + v_*(t, u)\} \times [0, 2^{-t}] \\ K_{t,u}^1 &= [c_*(t, u) - v_*(t, u), c_*(t, u) + v_*(t, u)] \times \{2^{-t}\} \\ K_{t,u}^2 &= (c_*(t, u) - v_*(t, u), c_*(t, u) + v_*(t, u)) \times (-1, 2^{-t}) \\ K_t &= \left(K_t^* \cup \bigcup_{i < 2} \bigcup_{u \in \mathbb{N}} K_{t,u}^i \right) \cup \left(([2^{-(2t+1)}, 2^{-2t}] \times \{0\}) \setminus \bigcup_{u \in \mathbb{N}} K_{t,u}^2 \right). \end{aligned}$$

Note that K_t is homeomorphic to the harmonic comb \mathcal{H} for each $t \in \mathbb{N}$. This is because, for fixed $t \in \mathbb{N}$, since $\lim_s f_s(t)$ exists we have $v(t, u) = 0$ for almost all $u \in \mathbb{N}$. Then the desired computable dendroid is defined as follows.

$$K = ([-1, 0] \times \{0\}) \cup \bigcup_{t \in \mathbb{N}} \left(([2^{-(2t+2)}, 2^{-(2t+1)}] \times \{0\}) \cup K_t^- \right).$$

Claim. The set K is a computable dendroid.

Clearly K is a computable closed set. To show that K is pathwise connected, we consider the following subcontinuum K_t^- , the grip of the comb $K_{t,m}$, for each $t \in \mathbb{N}$.

$$K_t^- = \left(\bigcup_{i < 2} \bigcup_{v(t, u) > 0} K_{t,u}^i \right) \cup \left(([2^{-(2t+1)}, 2^{-2t}] \times \{0\}) \setminus \bigcup_{v(t, u) > 0} K_{t,u}^2 \right).$$

Then $K^- = ([-1, 0] \times \{0\}) \cup \bigcup_{t \in \mathbb{N}} \left(([2^{-(2t+2)}, 2^{-(2t+1)}] \times \{0\}) \cup K_t^- \right)$ has no ramification points. We claim that K^- is connected and K^- is even an arc. To show this claim, we first observe that K_t^- is an arc for any $t \in \mathbb{N}$, since $v(t, u) > 0$ occurs for finitely many $u \in \mathbb{N}$. Moreover $K_t^- \subseteq S(t)$, and $\lim_t \text{diam}(S(t)) = 0$ holds. Therefore, we see that K^- is locally connected and, hence, an arc. For points $p, q \in K$, if $p, q \in K^-$ then p and q are connected by a subarc of K^- . In the case $p \in K \setminus K^-$, the point p lies on $K_{t,u}^0$ for some t, u such that $v(t, u) = 0$. If $q \in K^-$ then there is a subarc $A \subseteq K^-$ (one of whose endpoints must be $\langle c_*(t, u), 0 \rangle$) such that $A \cup K_{t,u}^0$ is an arc containing p and q . In the case $q \in K \setminus K^-$, similarly we can connect p and q by an arc in K . Hence, K is

pathwise connected. K is hereditarily unicoherent, since the harmonic comb is hereditarily unicoherent. Thus, K is a dendroid.

Claim. The computable dendroid K does not $*$ -include a Π_1^0 dendrite.

What remains to show is that every Π_1^0 subdendrite $R \subseteq K$ satisfies $d_H(R, K) \geq 1$. Let $R \subseteq K$ be a Π_1^0 dendrite. Put $S(t) = [2^{-(2t+1)}, 2^{-2t}] \times [0, 2^{-t}]$. Since R is locally connected, $R \cap S(t) = R \cap K_t$ is also locally connected for each $t \in \mathbb{N}$ and $m < 2^t$. Thus, for fixed $t \in \mathbb{N}$, let $K_{t,u}^{1*} = [c_*(t, u) - 2^{-(2t+u+3)}, c_*(t, u) + 2^{-(2t+u+3)}] \times \{2^{-t}\}$. For any continuum $R^* \subset K_t$, if $R^* \cap K_{t,u}^{1*} \neq \emptyset$ for infinitely many $u \in \mathbb{N}$, then R^* must be homeomorphic to the harmonic comb \mathcal{H} . Hence, R^* is not locally connected. Therefore, we have $R \cap K_{t,u}^{1*} = \emptyset$ for almost all $u \in \mathbb{N}$. Since $K_{t,u}^{1*}$ and $K_{s,v}^{1*}$ is disjoint whenever $\langle t, u \rangle \neq \langle s, v \rangle$, and since R is Π_1^0 , we can effectively enumerate $U_t = \{u \in \mathbb{N} : R \cap K_{t,u}^{1*} = \emptyset\}$, i.e., $\{U_t : t \in \mathbb{N}\}$ is uniformly c.e. Moreover, U_t is cofinite for every $t \in \mathbb{N}$. Then, by our definition of $f = \lim_s f_s$ in Lemma 12, there exists $t^* \in \mathbb{N}$ such that $f(t) \in U_t$ for all $t \geq t^*$. Note that $v(t, f(t)) > 0$ and thus the condition $f(t) \in U_t$ (i.e., $R \cap K_{t,f(t)}^{1*} = \emptyset$) implies that, for every $t \geq t^*$, either $R \subseteq [-1, c_*(t, u) + v_*(t, u)] \times [0, 1]$ or $R \subseteq [c_*(t, u) - v_*(t, u), 1] \times [0, 1]$ holds. Thus we obtain the desired condition $d_H(R, K) \geq 1$. \square

Remark. It is easy to see that the dendroid constructed in the proof of Theorem 11 is contractible.

Corollary 3. *There exists a nonempty contractible planar computable closed subset of $[0, 1]^2$ which $*$ -includes no Π_1^0 subset which is connected and locally connected.*

Theorem 13. *Not every nonempty Π_1^0 planar dendroid contains a computable point.*

Proof. One can easily construct a Π_1^0 Cantor fan F containing at most one computable point $p \in F$, and such p is the unique ramification point of F . Our basic idea is to construct a topological copy of the Cantor fan F along a pathological located arc A constructed by Miller [10, Example 4.1]. We can guarantee that moving the fan F along the arc A cannot introduce new computable points. Additionally, this moving will make a ramification point p^* in a copy of F incomputable.

Fat Approximation. To archive this construction, we consider a fat approximation of a subset P of the middle third Cantor set $C \subseteq \mathbb{R}^1$, by modifying the standard construction of C . For a tree $T \subseteq 2^{<\mathbb{N}}$, put $\pi(\sigma) = 3^{-1+2\sum_{i<lh(\sigma)} \& \sigma(i)=1} 3^{-(i+2)}$ for $\sigma \in T$, and $J(\sigma) = [\pi(\sigma) - 3^{-(lh(\sigma)+1)}, \pi(\sigma) + 2 \cdot 3^{-(lh(\sigma)+1)}]$. If a binary string σ is incomparable with a binary string τ then $J(\sigma) \cap J(\tau) = \emptyset$. We extend π to a homeomorphism π_* between Cantor space $2^{\mathbb{N}}$ and $C \cap [1/3, 2/3]$ by defining $\pi_*(f) = 3^{-1} + 2\sum_{f(i)=1} 3^{-(i+2)}$ for $f \in 2^{\mathbb{N}}$. Let $P^* \subseteq 2^{\mathbb{N}}$ be a nonempty Π_1^0 set without computable elements. Then there exists a computable tree T_P such that P^* is the set of all paths of T_P , since P^* is Π_1^0 . A *fat approximation* $\{P_s : s \in \mathbb{N}\}$ of $P = \pi_*(P^*)$ is defined as $P_s = \bigcup \{J(\sigma) : lh(\sigma) = s \& \sigma \in T_P\}$. Then $\{P_s : s \in \mathbb{N}\}$ is a computable decreasing sequence of computable closed sets, and we have $P = \bigcap_s P_s$. Since P is a nonempty bounded closed subset of a real line \mathbb{R}^1 , both $\min P$

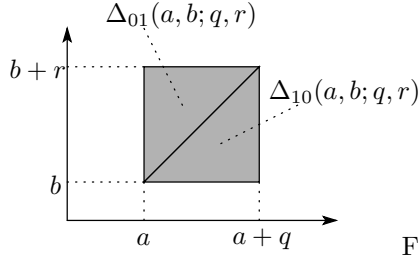


Figure 11: The cubes $\Delta_{ij}(a, b, q, r)$. $[1/6, 1/2]$

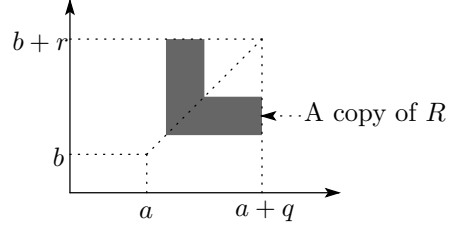


Figure 12: $[\sqcup](R; a, b, q, r)$ for $R =$

and $\max P$ exist. By the same reason, both $l_s^- = \min P_s$ and $r_s^+ = \max P_s$ also exist, for each $s \in \mathbb{N}$, and $\lim_s l_s^- = \min P$ and $\lim_s r_s^+ = \max P$, where $\{l_s : s \in \mathbb{N}\}$ is increasing, and $\{r_s : s \in \mathbb{N}\}$ is decreasing. Let $l_s = l_s^- + 3^{-(s+1)}$ and $r_s = r_s^+ - 3^{-(s+1)}$. We also set $l_s^* = l_s^- + 3^{-(s+2)}$ and $r_s = r_s^+ - 3^{-(s+2)}$. Note that $l_s < r_s$, $\lim_s l_s = \min P$, and $\lim_s r_s = \max P$. Since $\min P, \max P \in P$ and P contains no computable points, $\min P$ and $\max P$ are non-computable, and so $l_s < \min P < \max P < r_s$ holds for any $s \in \mathbb{N}$. The fat approximation of P has the following remarkable property:

$$[l_s^-, l_s] \subseteq P_s, [l_s^-, l_s] \cap P = \emptyset, [r_s, r_s^+] \subseteq P_s, \text{ and } [r_s, r_s^+] \cap P = \emptyset.$$

To simplify the construction, we may also assume that P has the following property:

$$P = \{1 - x \in \mathbb{R} : x \in P\}$$

Because, for any Π_1^0 subset $A \subseteq C$, the Π_1^0 set $A^* = \{x/3 : x \in A\} \cup \{1 - x/3 : x \in A\} \subseteq C$ has that property.

Basic Notation. For each $i, j < 2$, for each $a, b \in \mathbb{R}^2$, and for each $q, r \in \mathbb{R}$, the 2-cube $\Delta_{ij}(a, b; q, r) \subseteq [a, a+q] \times [b, b+r]$ is defined as the smallest convex set containing the three points $\{(a, b), (a+q, b), (a, b+r), (a+q, b+r)\} \setminus \{(a+(1-i)q, b+(1-j)r)\}$. Namely,

$$\Delta_{ij}(a, b; q, r) = \{((-1)^i x + a + iq, (-1)^j y + b + jr) \in \mathbb{R}^2 : x, y \geq 0 \text{ \& } rx + qy \leq qr\}.$$

For a set $R \subseteq \mathbb{R}^1$ and real numbers $r, y \in \mathbb{R}$, put $\Theta(R; r, y) = \{rx + y \in \mathbb{R} : x \in R\}$. Clearly $\Theta(R; r, y)$ is computably homeomorphic to R . Let four symbols $\sqcup, \sqcap, \sqcup, \sqcap$ denote $\langle 10, 01 \rangle, \langle 01, 10 \rangle, \langle 00, 11 \rangle$, and $\langle 11, 00 \rangle$, respectively. For $v \in \{\sqcup, \sqcap, \sqcup, \sqcap\}$ and for any $R \subseteq [0, 1]$, $a, b \in \mathbb{R}^2$, and $q, r \in \mathbb{R}$, we define $[v](R; a, b; q, r) \subseteq [a, a+q] \times [b, b+r]$ as follows:

$$\begin{aligned} [v](R; a, b; q, r) = & (([a, a+q] \times \Theta(R; r, b)) \cap \Delta_{v(0)}(a, b; q, r)) \\ & \cup ((\Theta(R; q, a) \times [b, b+r]) \cap \Delta_{v(1)}(a, b; q, r)). \end{aligned}$$

Sublemma 1. $[v](P; a, b; q, r)$ is computably homeomorphic to $P \times [0, 1]$. In particular, $[v](P; a, b; q, r)$ contains no computable points.

To simplify our argument, we use the normalization \tilde{P}_t^s of P_t for $t \geq s$, that is defined by $\tilde{P}_t^s = \{(x - l_s^-)/(r_s^+ - l_s^-) \in \mathbb{R} : x \in P_t\}$, for each $s \in \mathbb{N}$.

Note that $\tilde{P}_t^s \subseteq [0, 1]$ for $t \geq s$, and $0, 1 \in \tilde{P}_s^s$ holds for each $s \in \mathbb{N}$. Put $[v]_t^s([a, a+q] \times [b, b+r]) = [v](\tilde{P}_t^s; a, b; q, r)$ for $t \geq s$. We also introduce the following two notions:

$$\begin{aligned} [-]_t^s([a, a+q] \times [b, b+r]) &= [a, a+q] \times \Theta(\tilde{P}_t^s; r, b); \\ [|]_t^s([a, a+q] \times [b, b+r]) &= \Theta(\tilde{P}_t^s; q, a) \times [b, b+r]. \end{aligned}$$

Here we code two symbols $-$ and $|$ as 0 and 1 respectively.

Sublemma 2. $[v]_t^s([a, a+q] \times [b, b+r]) \subseteq [a, a+q] \times [b, b+r]$, and $[v]_t^s([a, a+q] \times [b, b+r])$ intersects with the boundary of $[a, a+q] \times [b, b+r]$.

Sublemma 3. There is a computable homeomorphism between $[v]_t^s(a, b; q, r)$ and $P_t \times [0, 1]$ for any $t \in \mathbb{N}$. Therefore, $\bigcap_t [v]_t^s(a, b; q, r)$ is computably homeomorphic to $P \times [0, 1]$.

Blocks. A block is a set $Z \subseteq \mathbb{R}^2$ with a bounding box $\text{Box}(Z) = [a, a+q] \times [b, b+r]$. Each $\delta \in 2^2$ is called a *direction*, and directions $\langle 00 \rangle$, $\langle 01 \rangle$, $\langle 10 \rangle$, and $\langle 11 \rangle$ are also denoted by $[\leftarrow]$, $[\rightarrow]$, $[\downarrow]$, and $[\uparrow]$, respectively. For $\delta \in 2^2$, $\delta^\circ = \langle \delta(0), 1 - \delta(0) \rangle$ is called the *reverse direction* of δ . Put $\text{Line}(Z; [\leftarrow]) = \{a\} \times [b, b+r]$; $\text{Line}(Z; [\rightarrow]) = \{a+q\} \times [b, b+r]$; $\text{Line}(Z; [\downarrow]) = [a, a+q] \times \{b\}$; $\text{Line}(Z; [\uparrow]) = [a, a+q] \times \{b+r\}$. Assume that a class \mathcal{Z} of blocks is given. We introduce a relation $\overset{\delta}{\dashrightarrow}$ on \mathcal{Z} for each direction δ . Fix a block $Z_{\text{first}} \in \mathcal{Z}$, and we call it the *first block*. Then we declare that $\overset{[\leftarrow]}{\dashrightarrow} Z_{\text{first}}$ holds. We inductively define the relation $\overset{\delta}{\dashrightarrow}$ on \mathcal{Z} . If $Z \overset{\delta}{\dashrightarrow} Z_0$ (resp. $Z_0 \overset{\delta}{\dashrightarrow} Z$) for some Z and δ , then we also write it as $\overset{\delta}{\dashrightarrow} Z_0$ (resp. $Z_0 \overset{\delta}{\dashrightarrow}$). For any two blocks Z_0 and Z_1 , the relation $Z_0 \overset{\delta}{\dashrightarrow} Z_1$ holds if the following three conditions are satisfied:

1. $Z_0 \cap Z_1 = \text{Line}(Z_0; \delta) \cap Z_0 = \text{Line}(Z_1; \delta^\circ) \cap Z_1 \neq \emptyset$.
2. $\overset{\varepsilon}{\dashrightarrow} Z_0$ has been already satisfied for some direction ε .
3. $Z_1 \overset{\varepsilon}{\dashrightarrow} Z_0$ does not satisfied for any direction ε

If $Z_0 \overset{\delta}{\dashrightarrow} Z_1$ for some δ , then we say that Z_1 is a *successor* of Z_0 (Z_0 is a predecessor of Z_1), and we also write it as $Z_0 \dashrightarrow Z_1$.

We will construct a partial computable function $\mathcal{Z} : \mathbb{N}^3 \rightarrow \mathcal{A}(\mathbb{R}^2)$ with a computable function $k : \mathbb{N} \rightarrow \mathbb{N}$ and $\text{dom}(\mathcal{Z}) = \{(u, i, t) \in \mathbb{N}^3 : u \leq t \text{ \& } i < k(u)\}$ such that $\mathcal{Z}(u, i, t)$ is a block with a bounding box for any $(u, i, t) \in \text{dom}(\mathcal{Z})$, and the block $\mathcal{Z}(u, i, t)$ is computably homeomorphic to $P_t \times [0, 1]$ uniformly in (u, i, t) . Here $\mathcal{A}(\mathbb{R}^2)$ is the hyperspace of all closed subsets in \mathbb{R}^2 with positive and negative information. For each stage t , $\mathcal{Z}_t(u) = \{\mathcal{Z}(t, u, i) : i < k(u)\}$ for each $u \leq t$ is defined. Let $\mathcal{Z}(u)$ denote the finite set $\{\lambda t. \mathcal{Z}(t, u, i) : i < k(u)\}$ of functions, for each $u \in \mathbb{N}$. The relation \dashrightarrow induces a pre-ordering \prec on $\bigcup_{u \in \mathbb{N}} \mathcal{Z}(u)$ as follows: $Z_0 \prec Z_1$ if there is a finite path from $Z_0(t)$ to $Z_1(t)$ on the finite directed graph $(\bigcup_{u \leq t} \mathcal{Z}_t(u), \dashrightarrow)$ at some stage $t \in \mathbb{N}$. We will assure that \prec is a well-ordering of order type ω , and $Z_0 \prec Z_1$ whenever $Z_0 \in \mathcal{Z}(u)$, $Z_1 \in \mathcal{Z}(v)$, and $u < v$. In particular, for every $Z \in \bigcup_{u \in \mathbb{N}} \mathcal{Z}(u)$, the predecessor Z_{pre} of Z and the successor Z_{suc} of Z under \prec are uniquely determined. If $Z_{\text{pre}}(t) \overset{\delta}{\dashrightarrow} Z(t) \overset{\varepsilon}{\dashrightarrow} Z_{\text{suc}}(t)$, then we say that Z moves from δ to ε , and that $\langle \delta, \varepsilon \rangle$ is the *direction* of Z .

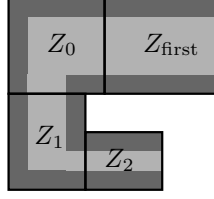


Figure 13: Example 14.

Example 14. Fig. 13 is an example satisfying $\xrightarrow{[\leftarrow]} Z_{\text{first}} \xrightarrow{[\leftarrow]} Z_0 \xrightarrow{[\downarrow]} Z_1 \xrightarrow{[\rightarrow]} Z_2$.

Destination Point. Basically, our construction is similar as the construction by Miller [10]. Pick the standard homeomorphism ρ between $2^{\mathbb{N}}$ and the middle third Cantor set, i.e., $\rho(M) = 2 \sum_{i \in M} (1/3)^{i+1}$ for $M \subseteq \mathbb{N}$, and pick a non-computable c.e. set $B \subseteq \mathbb{N}$ and put $\gamma = \rho(B)$. We will construct a Cantor fan so that the first coordinate of the unique ramification point is γ , hence the fan will have a non-computable ramification point. Let $\{B_s : s \in \mathbb{N}\}$ be a computable enumeration of B , and let n_s denote the element enumerated into B at stage s , where we may assume just one element is enumerated into B at each stage. Put $\gamma_s^{\min} = \rho(B_s)$ and $\gamma_s^{\max} = \rho(B_s \cup \{i \in \mathbb{N} : i \geq n_s\})$. Note that γ is not computable, and so $\gamma_s^{\min} \neq \gamma$ and $\gamma_s^{\max} \neq \gamma$ for any $s \in \mathbb{N}$. This means that for every $s \in \mathbb{N}$ there exists $t > s$ such that $\gamma_s^{\min} \neq \gamma_t^{\min}$ and $\gamma_s^{\max} \neq \gamma_t^{\max}$. By this observation, without loss of generality, we can assume that $\gamma_s^{\min} \neq \gamma_t^{\min}$ and $\gamma_s^{\max} \neq \gamma_t^{\max}$ whenever $s \neq t$. We can also assume $1/3 \leq \gamma_s^{\min} \leq \gamma_s^{\max} \leq 2/3$ for any $s \in \mathbb{N}$.

Stage 0. We now start to construct a Π_1^0 Cantor fan $Q = \bigcap_{s \in \mathbb{N}} Q_s$. At the first stage 0, and for each $t \geq 0$, we define the following sets:

$$Z_{0,t}^{\text{st}} = [-]_t^s([\gamma_0^{\min}, \gamma_0^{\max}] \times [l_0^-, r_0^+]); \quad Z_0^{\text{end}} = [\gamma_0^{\min} - 1/3, \gamma_0^{\min}] \times [l_0^-, r_0^+].$$

Moreover, we set $Q_0 = Z_{0,0}^{\text{st}} \cup Z_0^{\text{end}}$. By our choice of P_0 , actually $Q_0 = [\gamma_0^{\min} - 1/3, \gamma_0^{\max}] \times [l_0^-, r_0^+]$. $Z_{0,0}^{\text{st}}$ is called *the straight block from 2/3 to 1/3 at stage 0*, and Z_0^{end} is called *the end box at stage 0*. The *bounding box* of the block Z_0^{st} is defined by $[\gamma_0^{\min}, \gamma_0^{\max}] \times [l_0^-, r_0^+]$. The *collection of 0-blocks at stage t* is $\mathcal{Z}_t(0) = \{Z_{0,t}^{\text{st}}\}$. We declare that Z_0^{st} is the first block, and that $\xrightarrow{[\leftarrow]} Z_0^{\text{st}}$.

Stage s+1. Inductively assume that we have already constructed the collection of u -blocks $\mathcal{Z}_t(u)$ at stage $t \geq u$ is already defined for every $u \leq s$. For any u , we think of the collection $\mathcal{Z}(u) = \{\mathcal{Z}_t(u) : t \geq u\}$ as a finite set $\{Z_i^u\}_{i < \#\mathcal{Z}_u(u)}$ of computable functions $Z_i^u : \{t \in \mathbb{N} : t \geq u\} \rightarrow \bigcup_t \mathcal{Z}_t(u)$ such that $\mathcal{Z}_t(u) = \{Z_i^u(t) : i < \#\mathcal{Z}_u(u)\}$ for each $t \geq u$. We inductively assume that the collection $\mathcal{Z}(u) = \{\mathcal{Z}_t(u) : t \geq u\}$ satisfies the following conditions:

- (IH1) For each $Z \in \mathcal{Z}(u)$ and for each $t \geq v \geq u$, $Z(t) \subseteq Z(v)$.
- (IH2) There is a computable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f \upharpoonright \bigcup_{u \leq s} \mathcal{Z}_t(u)$ is a homeomorphism between $\bigcup_{u \leq s} \mathcal{Z}_t(u)$ and $P_t \times [0, 1]$ for any $t \geq s$.

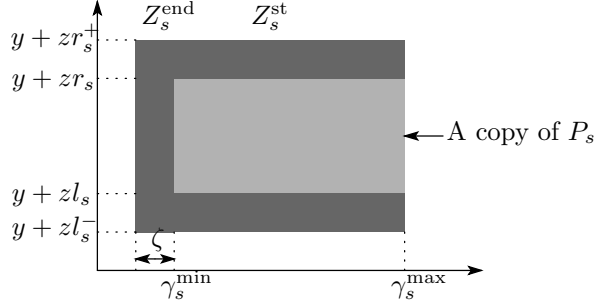


Figure 14: The active block $Z_s^{\text{st}} \cup Z_s^{\text{end}}$ at stage s .

(IH3) There are $y, z, \zeta \in \mathbb{Q}$ such that the blocks $Z_{s,t}^{\text{st}}$ and Z_s^{end} are constructed as follows:

$$\begin{aligned} Z_{s,t}^{\text{st}} &= [-]_t^s([\gamma_s^{\min}, \gamma_s^{\max}] \times [y + zl_s^-, y + zr_s^+]); \\ Z_s^{\text{end}} &= [\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zl_s^-, y + zr_s^+]. \end{aligned}$$

Here, a computable closed set Q_s , an approximation of our Π_1^0 Cantor fan Q at stage s , is defined by $Q_s = Z_s^{\text{end}} \cup \bigcup_{u \leq s} Z_s(u)$.

Non-injured Case. First we consider the case $[\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \subseteq [\gamma_s^{\min}, \gamma_s^{\max}]$, i.e., this is the case that our construction is *not injured* at stage $s+1$. In this case, we construct $(s+1)$ -blocks in the active block $Z_s^{\text{st}} \cup Z_s^{\text{end}}$. We will define $Z_t(s, i, j)$ and $\text{Box}(s, i, j) = \text{Box}(Z_t(s, i, j))$ for each $j < 6$. The first two corner blocks at stage $t \geq s+1$ are defined by:

$$\begin{aligned} \text{Box}(s, 0) &= [\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zl_s^-, y + zr_s^*], \\ Z_t(s, 0) &= [\downarrow]_t^s([\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zl_s^-, y + zr_s^*]) \cap \text{Box}(s, 0), \\ \text{Box}(s, 1) &= [\gamma_s^{\min} - \zeta, \gamma_s^{\min}] \times [y + zr_s^*, y + zr_s^+], \\ Z_t(s, 1) &= [\uparrow]_t^s(\text{Box}(s, 1)). \end{aligned}$$

Sublemma 4. $Z_t(s, 0) \cup Z_t(s, 1) \subseteq Z_s^{\text{end}}$ for any $t \geq s+1$.

Sublemma 5. $Z_{s,t}^{\text{st}} \xrightarrow{[\leftarrow]} Z_t(s, 0) \xrightarrow{[\uparrow]} Z_t(s, 1)$, for any $t \geq s+1$.

The next block is a straight block from γ_s^{\min} to γ_{s+1}^{\max} which is defined as follows:

$$\begin{aligned} \text{Box}(s, 2) &= [\gamma_s^{\min}, \gamma_s^{\max}] \times [y + zr_s^*, y + zr_s^+], \\ Z_t(s, 2) &= [-](\text{Box}(s, 2)). \end{aligned}$$

For given $a, b, \alpha, \beta \in \mathbb{Q}$, we can calculate $N_{0,s}(a, b; \alpha, \beta)$ and $N_{1,s}(a, b; \alpha, \beta)$ satisfying $N_{0,s}(a, b; \alpha, \beta) + N_{1,s}(a, b; \alpha, \beta) \cdot l_s^- = a + b\alpha$, and $N_{0,s}(a, b; \alpha, \beta) + N_{1,s}(a, b; \alpha, \beta) \cdot r_s^+ = a + b\beta$. Put $y^* = N_{0,s}(y, z; r_s^*, r_s^+)$, and $z^* = N_{1,s}(y, z; r_s^*, r_s^+)$.

Sublemma 6. $\text{Box}(s, 2) = [\gamma_s^{\min}, \gamma_s^{\max}] \times [y^* + z^*l_s^-, y^* + z^*r_s^+]$.

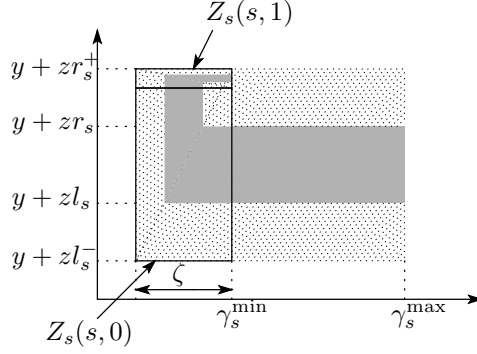


Figure 15: The first two corner blocks $Z_s(s, 0)$ and $Z_s(s, 1)$.

Put $\zeta^* = (\gamma_s^{\max} - \gamma_{s+1}^{\max})/3^s$. Note that $\zeta^* > 0$ since $\gamma_s^{\max} > \gamma_{s+1}^{\max}$. We then again define *corner blocks*.

$$\begin{aligned} \text{Box}(s, 3) &= [\gamma_{s+1}^{\max}, \gamma_{s+1}^{\max} + \zeta^*] \times [y^* + z^* l_s^-, y^* + z^* r_s^*], \\ Z_t(s, 3) &= [-]_t^s([\gamma_{s+1}^{\max}, \gamma_{s+1}^{\max} + \zeta^*] \times [y^* + z^* l_s^-, y^* + z^* r_s^*]) \cap \text{Box}(s, 3), \\ \text{Box}(s, 4) &= [\gamma_{s+1}^{\max}, \gamma_{s+1}^{\max} + \zeta^*] \times [y^* + z^* r_s^*, y^* + z^* r_s^+], \\ Z_t(s, 4) &= [-]_t^s(\text{Box}(s, 4)). \end{aligned}$$

Next, a *straight block* from γ_s^{\min} to γ_{s+1}^{\max} is defined as follows:

$$\begin{aligned} \text{Box}(s, 5) &= [\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \times [y^* + z^* r_s^*, y^* + z^* r_s^+], \\ Z_t(s, 5) &= [-]_t^s[\text{Box}(s, 5)]. \end{aligned}$$

Put $y^{**} = N_{0,s}(y^*, z^*; r_s^*, r_s^+)$, and $z^{**} = N_{1,s}(y^*, z^*; r_s^*, r_s^+)$.

Sublemma 7. $\text{Box}(s, 5) = [\gamma_s^{\min}, \gamma_s^{\max}] \times [y^{**} + z^{**} l_s^-, y^{**} + z^{**} r_s^+]$.

Put $\zeta^{**} = (\gamma_{s+1}^{\min} - \gamma_s^{\min})/3^s$. Note that $\zeta^{**} > 0$ since $\gamma_{s+1}^{\min} > \gamma_s^{\min}$. The end box at stage $s+1$ is:

$$Z(s, 6) = [\gamma_{s+1}^{\min} - \zeta^{**}, \gamma_{s+1}^{\min}] \times [y^{**} + z^{**} l_s^-, y^{**} + z^{**} r_s^+].$$

Then put $Z_{s+1,t}^{\text{st}} = Z_t(s, 5)$, $Z_{s+1}^{\text{st}} = Z_{s+1,s+1}^{\text{st}}$, and $Z_{s+1}^{\text{end}} = Z(s, 6)$. The *active block* at stage $s+1$ is the set $Z_{s+1,s+1}^{\text{st}} \cup Z_{s+1}^{\text{end}}$, and the *collection of $(s+1)$ -blocks* at stage t is defined by $Z_t(s+1) = \{Z_t(s, i) : i \leq 5\}$. Clearly, our definition satisfies the induction hypothesis (IH3) at stage $s+1$.

Sublemma 8. $Z_t(s, i) \subseteq Z_v(s, i)$ for each $t \geq v \geq s+1$ and $i \leq 5$.

Sublemma 9. For any $t \geq s+1$,

$$Z_{s,t}^{\text{st}} \xrightarrow{[\leftarrow]} Z_t(s, 0) \xrightarrow{[\uparrow]} Z_t(s, 1) \xrightarrow{[\rightarrow]} Z_t(s, 2) \xrightarrow{[\rightarrow]} Z_t(s, 3) \xrightarrow{[\uparrow]} Z_t(s, 4) \xrightarrow{[\leftarrow]} Z_t(s, 5).$$

Proof. It follows straightforwardly from the definition of these blocks $Z_t(s, i)$, and Sublemma 6 and 7. \square

Sublemma 10. $\bigcup_{2 \leq i \leq 6} Z_t(s, i) \subseteq Z_s^{\text{st}} \cap [\gamma_s^{\min}, \gamma_s^{\max}] \times (y + z r_s, y + z r_s^+]$. Hence, $(\bigcup_{2 \leq i \leq 6} Z_t(s, i)) \cap Z_{s,s+1}^{\text{st}} = \emptyset$

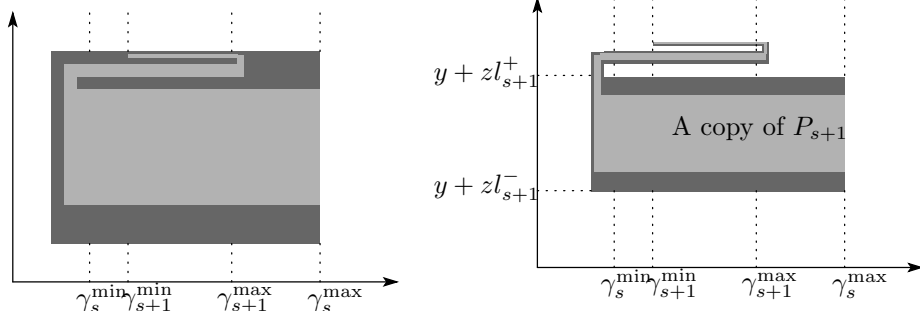


Figure 16: $Z_s(s-1, 5) \cup \bigcup_{u \leq s} Z_s(s+1)$. Figure 17: $Z_{s+1}(s-1, 5) \cup \bigcup_{u \leq s+1} Z_{s+1}(s+1)$.

Consequently, we can show the following extension property.

Sublemma 11. *Assume that we have a computable function $f_s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f_s \upharpoonright \bigcup_{u \leq s} Z_t(u)$ is a computable homeomorphism between $\bigcup_{u \leq s} Z_t(u)$ and $P_t \times [1/(s+2), 1]$ for any $t \geq s$. Then we can effectively find a computable function $f_{s+1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ extending $f_s \upharpoonright \bigcup_{u \leq s} Z_{s+1}(u)$ such that $f_{s+1} \upharpoonright \bigcup_{u \leq s+1} Z_t(u)$ is a computable homeomorphism between $\bigcup_{u \leq s+1} Z_t(u)$ and $P_t \times [1/(s+3), 1]$ for any $t \geq s+1$.*

Proof. By Sublemma 5, 9, and 10. \square

By Sublemma 8 and 11, induction hypothesis (IH1) and (IH2) are satisfied. Since $Z_{s+1}^{\text{end}} \cup \bigcup_{u \leq s+1} Z_{s+1}(s+1) \subseteq Z_s^{\text{st}} \cup Z_s^{\text{end}}$ by Sublemma 4 and 10, and $\bigcup_{u \leq s+1} Z_{s+1}(u) \subseteq \bigcup_{u \leq s} Z_s(u)$ for each $u \leq s$, by induction hypothesis (IH1), we have the following:

$$Q_{s+1} = Z_{s+1}^{\text{end}} \cup \bigcup_{u \leq s+1} Z_{s+1}(u) \subseteq Z_s^{\text{st}} \cup Z_s^{\text{end}} \cup \bigcup_{u \leq s} Z_s(u) \subseteq Q_s.$$

Injured Case. Secondly we consider the case that our construction is *injured*. This means that $[\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \not\subseteq [\gamma_s^{\min}, \gamma_s^{\max}]$. In this case, indeed, we have $[\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \cap [\gamma_s^{\min}, \gamma_s^{\max}] = \emptyset$. Fix the greatest stage $p \leq s$ such that $[\gamma_{s+1}^{\min}, \gamma_{s+1}^{\max}] \subseteq [\gamma_p^{\min}, \gamma_p^{\max}]$ occurs. We again, inside the end box Z_s^{end} at stage s , define corner blocks $Z_t(s, 0)$ and $Z_t(s, 1)$ as non-injuring stage, whereas the construction of $Z_t(s, i)$ for $i \geq 2$ differs from non-injuring stage. The end box of our construction at stage $s+1$ will turn back along all blocks belonging $Z_s(u)$ for $p < u \leq s$ in the reverse ordering of \prec . Let $\{Z_i : i < k_{s+1}\}$ be an enumeration of all blocks in $Z_s(u)$ for $p < u \leq s$, under the reverse ordering of \prec . In other words, Z_i is the successor block of Z_{i+1} under \rightarrow , for each $i < k_{s+1} - 1$. There are two kind of blocks; one is a *straight block*, and another is a *corner block*. We will define blocks $Z_t(s, i, j)$ for $i < k_{s+1}$ and $j < 3$. Now we check the direction $\langle \delta_i, \varepsilon_i \rangle$ of Z_i . Here, we may consistently assume that the condition $Z_0 \xrightarrow{[\leftarrow]}$ holds.

Subcase 1. If $\delta_i(0) = \varepsilon_i(0)$ then Z_i is a straight block. In this case, we only construct $Z_t(s, i, 0)$. Since Z_i is straight, there are $y_i, z_i, \alpha, \beta \in \mathbb{Q}$ and $u \leq s$ such that, for $B_i(0) = [\alpha, \beta]$ and $B_i(1) = [y_i + z_i l_u^-, y_i + z_i r_u^+]$ such that $\text{Box}(Z_i) = B_i(\delta_i(0)) \times B_i(1 - \delta_i(0))$. If $\delta_i(1) = 0$, then set $y_i^* = N_{0,s}(y_i, z_i; l_s^-, l_s^*)$ and $z_i^* = N_{1,s}(y_i, z_i; l_s^-, l_s^*)$. If $\delta_i(1) = 1$, then set $y_i^* = N_{0,s}(y_i, z_i; r_s^-, r_s^*)$ and

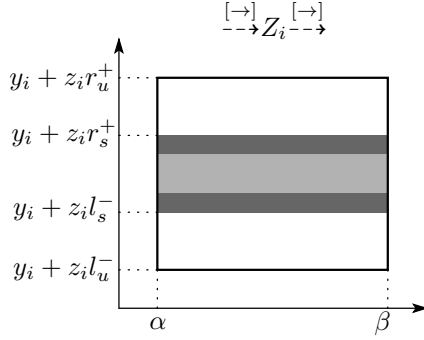


Figure 18: The block Z_i .

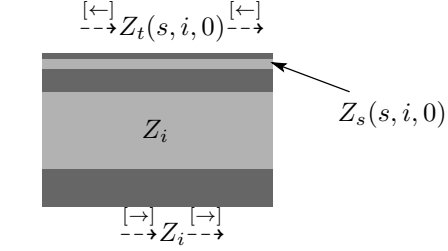


Figure 19: The block $Z_t(s, i, 0)$.

$z_i^* = N_{1,s}(y_i, z_i; r_s^*, r_s^+)$. Then, we define $Z_t(s, i, 0)$ as the following straight block:

$$\begin{aligned} B_i^*(0) &= B_i(0); \quad B_i^*(1) = [y_i^* + z_i^* l_s^-, y_i^* + z_i^* r_s^+]; \\ Z_t(s, i, 0) &= [\delta_i(0)]_t^s (B_i^*(\delta_i(0)) \times B_i^*(1 - \delta_i(0))). \end{aligned}$$

Here, $\text{Box}(Z_t(s, i, 0))$ is defined by $B_i^*(\delta_i(0)) \times B_i^*(1 - \delta_i(0))$.

Sublemma 12. $Z_t(s, i, 0) \subseteq Z_i$.

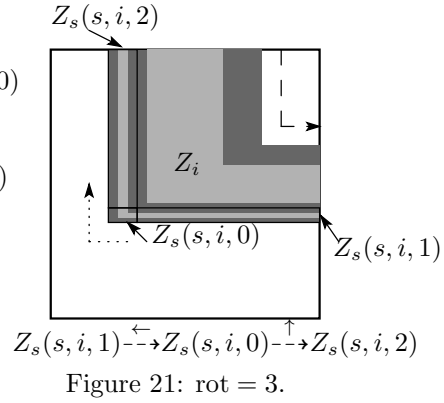
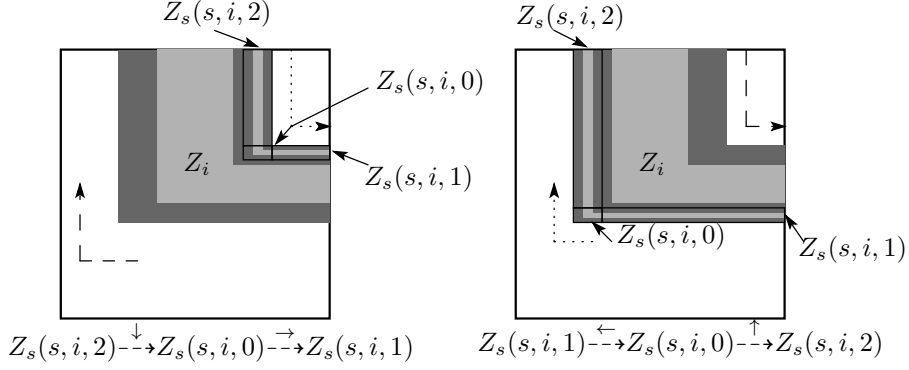
Proof. By our definition of $N_{0,s}$ and $N_{1,s}$, we have $B_i^*(1) = [y_i + z_i l_s^-, y_i + z_i l_s^*]$ or $B_i^*(1) = [y_i + z_i r_s^*, y_i + z_i r_s^+]$. \square

Subcase 2. If $\delta_i(0) \neq \delta_i(2)$ then Z_i is a corner block. We will construct three blocks; one corner block $Z_t(s, i, 0)$, and two straight blocks $Z_t(s, i, 1)$ and $Z_t(s, i, 2)$. We may assume that Z_i is of the following form:

$$\begin{aligned} Z_i &= [e]_s^u ([x_i + \zeta_i l_u^-, x_i + \zeta_i r_u^+] \times [y_i + z_i l_u^-, y_i + z_i r_u^+]), \\ \text{or } Z_i &= [e]_s^u ([x_i + \zeta_i l_u^-, x_i + \zeta_i r_u^+] \times [y_i + z_i l_u^-, y_i + z_i r_u^+]) \\ &\quad \cap ([x_i + \zeta_i l_u^-, x_i + \zeta_i r_u^+] \times [y_i + z_i l_u^-, y_i + z_i r_u^*]) \end{aligned}$$

Set $z = 0$ if the former case occurs; otherwise, set $z = 1$. Let $\{p_n : n < 6\}$ be an enumeration of $\{l_u^-, l_s^-, l_s^*, r_s^*, r_s^+, r_u^+\}$ in increasing order, and let p_6 be r_u^* . First we compute the value $\text{rot} = 2|\varepsilon_i(0) - |\delta_i(1) - \varepsilon_i(1)|| + 1$. Note that $\text{rot} \in \{1, 3\}$, and, if Z_i rotates clockwise then $\text{rot} = 1$; and if Z_i rotates counterclockwise then $\text{rot} = 3$. If $\xrightarrow{[->]} Z_i$ or $Z_i \xrightarrow{[->]}$, then put $D(0) = 1$; otherwise put $D(0) = 3$. If $\xrightarrow{[u]} Z_i$ or $Z_i \xrightarrow{[u]}$, then put $D(1) = 1$; otherwise put $D(1) = 3$. If $\xrightarrow{[->]} Z_i$ or $Z_i \xrightarrow{[->]}$, then put $E(0) = 0$; otherwise put $E(0) = 5 - \text{rot}$. If $\xrightarrow{[u]} Z_i$ or $Z_i \xrightarrow{[u]}$, then put $E(1) = 0$; otherwise put $E(1) = 5 - \text{rot}$. Then we now define $Z_t(s, i, j)$ for $j < 3$ as follows:

$$\begin{aligned} \text{Box}(s, i, 0) &= [x_i + \zeta_i p_{D(0)}, x_i + \zeta_i p_{D(0)+2}] \times [y_i + z_i p_{D(1)}, y_i + z_i p_{D(1)+2}], \\ \text{Box}(s, i, 1) &= [x_i + \zeta_i p_{E(0)}, x_i + \zeta_i p_{E(0)+\text{rot}}] \times [y_i + z_i p_{D(1)}, y_i + z_i p_{D(1)+2}], \\ \text{Box}(s, i, 2) &= [x_i + \zeta_i p_{D(0)}, x_i + \zeta_i p_{D(0)+2}] \times [y_i + z_i p_{E(1)}, y_i + z_i p_{E(1)+\text{rot}+z}], \\ Z_t(s, i, 0) &= [e]_t^s (\text{Box}(s, i, 0)), \\ Z_t(s, i, 1) &= [-]_t^s (\text{Box}(s, i, 1)), \\ Z_t(s, i, 2) &= [|]_t^s (\text{Box}(s, i, 2)). \end{aligned}$$



Intuitively, $D(0) = 1$ (resp. $D(0) = 3$) indicates that $Z_t(s, i, 0)$ passes the west (resp. the east) of Z_i ; $D(1) = 1$ (resp. $D(1) = 3$) indicates that $Z_t(s, i, 0)$ passes the south (resp. the north) of Z_i ; $E(0) = 0$ (resp. $E(0) = 5 - \text{rot}$) indicates that $Z_t(s, i, 1)$ passes the west (resp. the east) border of the bounding box of Z_i ; and $E(1) = 0$ (resp. $E(1) = 5 - \text{rot}$) indicates that $Z_t(s, i, 2)$ passes the south (resp. the north) border of the bounding box of Z_i . Note that the corner block $Z_t(s, i, 0)$ leaves Z_i on his right, and $Z_t(s, i, 0)$ has the reverse direction to Z_i .

Sublemma 13. $Z_t(s, i, 2 - \delta_i(0)) \xrightarrow{\varepsilon^\circ} Z_t(s, i, 0) \xrightarrow{\delta^\circ} Z_t(s, t, 1 + \delta_i(0))$.

Sublemma 14. $Z_t(s, i, j) \subseteq Z_i$.

For each $i < k_{s+1}$, we have already constructed $\mathcal{Z}_t(s+1; i) = \{Z_t(s, i, j) : j < 3\}$. All of these blocks constructed at the current stage are included in $Z_s^{\text{end}} \cup \bigcup_{p < u \leq s} \mathcal{Z}_s(u)$. Let $Z^0[i]$ (resp. $Z^1[i]$) be the \prec -least (resp. the \prec -greatest) element of $\{\lambda t. Z_t(s, i, j) : j < 3\}$. It is not hard to see that our construction ensures the following condition.

Sublemma 15. $Z_t^1[i] \dashrightarrow Z_t^0[i+1]$.

Thus, $\bigcup_{i < k_{s+1}} \mathcal{Z}_t(s+1; i)$ is computably homeomorphic to $P_t \times [0, 1]$, uniformly in $t \geq s+1$. Therefore, we can connect blocks $Z_s(s, i)$ for $i < k_{s+1}$, and we succeed to return back on the current approximation of the \prec -greatest p -block $Z_s(p) = Z_{p,s}^{\text{st}} \in \mathcal{Z}_s(p)$. Then we construct blocks $Z_t(s, k)$ for $2 \leq k \leq 6$ on the block $Z_s(p)$. The construction is essentially similar as the non-injuring case. By induction hypothesis (IH3), we note that $Z_s(p)$ must be of the following form for some $y_p, z_p \in \mathbb{Q}$:

$$Z_s(p) = [-]_s^p([\gamma_p^{\min}, \gamma_p^{\max}] \times [y_p + z_p l_p^-, y_p + z_p r_p^+]).$$

On $Z_s(p)$, we define a *straight block* from γ_p^{\min} to γ_{s+1}^{\max} as follows:

$$Z_t(s, 2) = [-]_s^p([\gamma_p^{\min}, \gamma_{s+1}^{\max}] \times [y_p + z_p r_s^*, y_p + z_p r_s^+]).$$

Here, by our assumption, $\gamma_{s+1}^{\max} < \gamma_p^{\max}$ holds since $\gamma_{s+1}^{\max} \leq \gamma_p^{\max}$. The blocks $Z_t(s, k)$ for $3 \leq k \leq 6$ are defined as in the same method as non-injuring case. The active block at stage $s+1$ is $Z_{s+1}(s, 5)$, and the end box at stage $s+1$

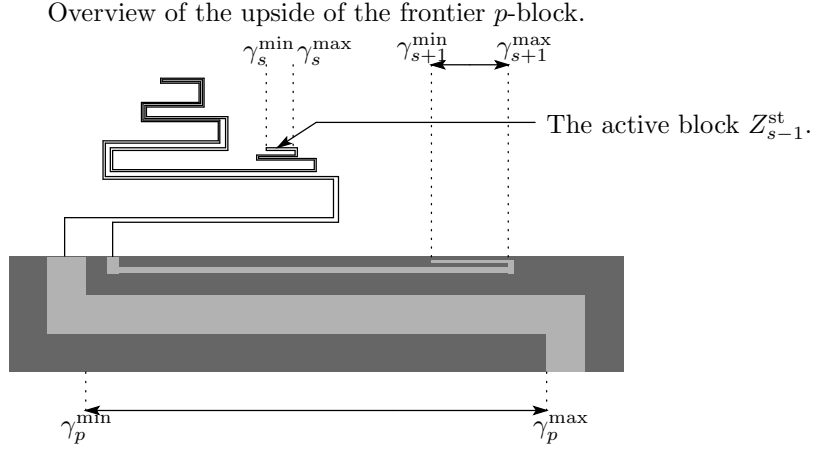


Figure 22: Outline of our construction of the injured case.

is $Z_{s+1}(s, 6)$. $(s+1)$ -blocks at stage t are $Z_t(s, i)$ for $i < 6$, and $Z_t(s, i, j)$ for $i < k_{s+1}$ and $j < 3$ if it is constructed. $\mathcal{Z}_t(s+1)$ denotes the collection of $(s+1)$ -blocks at stage t .

Sublemma 16. $Z_{s+1}^{\text{end}} \cup \bigcup \mathcal{Z}_{s+1}(s+1) \subseteq Z_s^{\text{end}} \cup \bigcup_{p \leq u \leq s} \mathcal{Z}_s(u)$.

Thus we again have the following:

$$Q_{s+1} = Z_{s+1}^{\text{end}} \cup \bigcup_{u \leq s+1} \mathcal{Z}_{s+1}(u) \subseteq Z_s^{\text{st}} \cup Z_s^{\text{end}} \cup \bigcup_{u \leq s} \mathcal{Z}_s(u) \subseteq Q_s.$$

Sublemma 17. Assume that we have a computable function $f_s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $f_s \upharpoonright \bigcup_{u \leq s} \mathcal{Z}_t(u)$ is a computable homeomorphism between $\bigcup_{u \leq s} \mathcal{Z}_t(u)$ and $P_t \times [1/(s+2), 1]$ for any $t \geq s$. Then we can effectively find a computable function $f_{s+1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ extending $f_s \upharpoonright \bigcup_{u \leq s} \mathcal{Z}_{s+1}(u)$ such that $f_{s+1} \upharpoonright \bigcup_{u \leq s+1} \mathcal{Z}_t(u)$ is a computable homeomorphism between $\bigcup_{u \leq s+1} \mathcal{Z}_t(u)$ and $P_t \times [1/(s+3), 1]$ for any $t \geq s+1$.

Finally we put $Q = \bigcap_{s \in \mathbb{N}} Q_s$ and $\mathcal{Z}^* = \bigcup_{u \in \mathbb{N}} \mathcal{Z}(u)$. The construction is completed.

Verification. Now we start to verify our construction.

Lemma 15. Q is Π_1^0 .

Sublemma 18. $\bigcap_{t \in \mathbb{N}} \bigcup_{Z \in \mathcal{Z}^*} Z_t = \bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t$.

Proof. The intersection $Z_s(p) \cap Z_s^i$ for $i < 2$ is included in some line segment $L_i \in \{[0, 1] \times \{b\}, \{b\} \times [0, 1] : b \in \mathbb{R}\}$, and $Z_s(p) \cap L_i = Z_s(p) \cap Z_s^i$ holds. \square

Sublemma 19. $\bigcup_{Z \in \mathcal{Z}(u)} \bigcap_{t \in \mathbb{N}} Z_t$ is computably homeomorphic to $[0, 1] \times P$, for each $u \in \mathbb{N}$.

Proof. By the induction hypothesis (IH2). \square

Sublemma 20. $\bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t$ is homeomorphic to $(0, 1] \times P$.

Proof. By Sublemma 11 and 17. \square

Lemma 16. *Q is homeomorphic to a Cantor fan.*

Proof. By Sublemma 18, there exists a real $y_0 \in \mathbb{R}$ such that the following holds:

$$Q = \left(\bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t \right) \cup \{ \langle \gamma, y_0 \rangle \}.$$

Therefore, by Sublemma 20, Q is homeomorphic to the one-point compactification of $(0, 1] \times P$. \square

Lemma 17. *Q contains no computable point.*

Proof. By Sublemma 19, $\bigcup_{Z \in \mathcal{Z}^*} \bigcap_{t \in \mathbb{N}} Z_t$ contains no computable point. \square

By Lemmata 15, 16, and 17, Q is the desired dendroid. \square

Remark. Since dendroids are compact and simply connected, Theorem 13 is the solution to the question of Le Roux and Ziegler [13]. Indeed, the dendroid constructed in the proof of Theorem 13 is contractible.

Corollary 4. *Not every nonempty contractible Π_1^0 subset of $[0, 1]^2$ contains a computable point.*

Question 18. *Does every locally connected planar Π_1^0 set contain a computable point?*

5 Immediate Consequences

5.1 Effective Hausdorff Dimension

For basic definition and properties of the effective Hausdorff dimension of a point of Euclidean plane, see Lutz-Weihrach [9]. For any $I \subseteq [0, 2]$, let DIM^I denote the set of all points in \mathbb{R}^2 whose effective Hausdorff dimensions lie in I . Lutz-Weihrach [9] showed that $\text{DIM}^{[1, 2]}$ is path-connected, but $\text{DIM}^{(1, 2]}$ is totally disconnected. In particular, $\text{DIM}^{(1, 2]}$ has no nondegenerate connected subset. It is easy to see that $\text{DIM}^{(0, 2]}$ has no nonempty Π_1^0 simple curve, since every Π_1^0 simple curve contains a computable point, and the effective Hausdorff dimension of each computable point is zero.

Theorem 19. *$\text{DIM}^{[1, 2]}$ has a nondegenerate contractible Π_1^0 subset.*

Proof. For any strictly increasing computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ with $f(0) = 0$ and any infinite binary sequence $\alpha \in 2^{\mathbb{N}}$, put $\iota_f(\alpha) = \prod_{i \in \mathbb{N}} \langle \alpha(i), \alpha(f(i)), \alpha(f(i)+1), \dots, \alpha(f(i+1)-1) \rangle$, where $\sigma \times \tau$ denotes the concatenation of binary strings σ and τ . Then, $r : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ is defined as $r(\alpha) = \sum_{i \in \mathbb{N}} (\alpha(i) \cdot 2^{-(i+1)})$. Note that $\alpha \neq \beta$ and $r(\alpha) = r(\beta)$ hold if and only if there is a common initial segment $\sigma \in 2^{<\mathbb{N}}$ of α and β such that $\sigma 0$ and $\sigma 1$ are initial segments of α and β respectively, and that $\alpha(m) = 1$ and $\beta(m) = 0$ for any $m > lh(\sigma)$, where $lh(\sigma)$ denotes the length of σ . In this case, we say that α sticks to β on σ . If $r(\alpha) \neq r(\beta)$, then clearly $r \circ \iota_f(\alpha) \neq r \circ \iota_f(\beta)$. Assume that α sticks to β on σ . Then there are $m_0 < m_1$ such that $\iota_f(\alpha)(m_0) = \iota_f(\alpha)(m_1) = \alpha(lh(\sigma)) = 0$ and $\iota_f(\beta)(m_0) = \iota_f(\beta)(m_1) = \beta(lh(\sigma)) = 1$ by our definition of ι_f . Therefore,

$\iota_f(\alpha)$ does not stick to $\iota_f(\beta)$. Hence, $r \circ \iota_f(\alpha) \neq r \circ \iota_f(\beta)$ whenever $\alpha \neq \beta$. Actually, $r \circ \iota_f : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ is a computable embedding. For each $n \in \mathbb{N}$, put $k_f(n) = \#\{s : f(s) < n\}$. Then, there is a constant $c \in \mathbb{N}$ such that, for any $\alpha \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have $K(\iota_f(\alpha) \upharpoonright n + k_f(n) + 1) \geq K(\alpha \upharpoonright n) - c$, where K denotes the prefix-free Kolmogorov complexity. Therefore, for any sufficiently fast-growing function $f : \mathbb{N} \rightarrow \mathbb{N}$ and any Martin-Löf random sequence $\alpha \in 2^{\mathbb{N}}$, the effective Hausdorff dimension of $r \circ \iota_f(\alpha)$ must be 1. Thus, for any nonempty Π_1^0 set $R \subseteq 2^{\mathbb{N}}$ consisting of Martin-Löf random sequences, $\{0\} \times (r \circ \iota_f(R))$ is a Π_1^0 subset of $\text{DIM}^{\{1\}}$. Let Q be the dendroid constructed from $P = r \circ \iota_f(R)$ as in the proof of Theorem 13, where we choose $\gamma = \rho(B)$ as Chaitin's halting probability Ω . For every point $\langle x_0, x_1 \rangle \in Q$, the effective Hausdorff dimension of x_i for some $i < 2$ is equivalent to that of an element of P or that of Ω . Consequently, $Q \subseteq \text{DIM}^{[1,2]}$. \square

5.2 Reverse Mathematics

Theorem 20. *For every Π_1^0 set $P \subseteq 2^{\mathbb{N}}$, there is a contractible planar Π_1^0 set Q such that Q is Turing-degree-isomorphic to P , i.e., $\{\deg_T(x) : x \in P\} = \{\deg_T(x) : x \in Q\}$.*

Proof. We choose B as a c.e. set of the same degree with the leftmost path of P . Then, the dendroid Q constructed from P and B as in the proof of Theorem 13 is the desired one. \square

A compact Π_1^0 subset P of a computable topological space is *Muchnik complete* if every element of P computes the set of all theorems of T for some consistent complete theory T containing Peano arithmetic. By Scott Basis Theorem (see Simpson [15]), P is Muchnik complete if and only if P is nonempty and every element of P computes an element of any nonempty Π_1^0 set $Q \subseteq 2^{\mathbb{N}}$.

Corollary 5. *There is a Muchnik complete contractible planar Π_1^0 set.*

A compact Π_1^0 subset P of a computable topological space is *Medvedev complete* (see also Simpson [15]) if there is a uniform computable procedure Φ such that, for any name $x \in \mathbb{N}^{\mathbb{N}}$ of an element of P , $\Phi(x)$ is the set of all theorems of T for some consistent complete theory T containing Peano arithmetic.

Question 21. *Does there exist a Medvedev complete simply connected planar Π_1^0 set? Does there exist a Medvedev complete contractible Euclidean Π_1^0 set?*

Our Theorem 13 also provides a reverse mathematical consequence. For basic notation for Reverse Mathematics, see Simpson [14]. Let RCA_0 denote the subsystem of second order arithmetic consisting of $I\Sigma_1^0$ (Robinson arithmetic with induction for Σ_1^0 formulas) and $\Delta_1^0\text{-CA}$ (comprehension for Δ_1^0 formulas). Over RCA_0 , we say that a sequence $(B_i)_{i \in \mathbb{N}}$ of open rational balls is *flat* if there is a homeomorphism between $\bigcup_{i < n} B_i$ and the open square $(0, 1)^2$ for any $n \in \mathbb{N}$. It is easy to see that RCA_0 proves that every flat cover of $[0, 1]$ has a finite subcover.

Theorem 22. *The following are equivalent over RCA_0 .*

1. *Weak König's Lemma: every infinite binary tree has an infinite path.*

2. Every open cover of $[0, 1]$ has a finite subcover.

3. Every flat open cover of $[0, 1]^2$ has a finite subcover.

Proof. The equivalence of the item (1) and (2) is well-known. It is not hard to see that RCA_0 proves the existence of the sequence $\{Q_s\}_{s \in \mathbb{N}}$ as in our construction of the dendroid Q in Theorem 13, by formalizing our proof in Theorem 13 in RCA_0 . Here we may assume that $\{Q_s\}_{s \in \mathbb{N}}$ is constructed from the set of all infinite paths of a given infinite binary tree $T \subseteq 2^{<\mathbb{N}}$, and a c.e. complete set $B \subseteq \mathbb{N}$. Note that $\bigcup_{s < t} ([0, 1]^2 \setminus Q_s)$ does not cover $[0, 1]^2$ for every $t \in \mathbb{N}$. Over RCA_0 , there is a flat sequence $\{[0, 1]^2 \setminus Q_s^*\}_{s \in \mathbb{N}}$ of open rational balls such that $\bigcap_{s < t} Q_s^* \supseteq \bigcap_{s < t} Q_s$ for any $t \in \mathbb{N}$, and that an open rational ball U is removed from some Q_s^* if and only if an open rational ball U is removed from some Q_u . However, if T has no infinite path, then Q has no element. In other words, $\{[0, 1]^2 \setminus Q_s^*\}_{s \in \mathbb{N}}$ covers $[0, 1]^2$. \square

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